THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and forty-third regular meeting of the Society was held in New York City on Saturday, April 24, 1909, extending through a morning and an afternoon session. About thirty-five persons were in attendance, including the following twenty-six members of the Society:

Professor Maxime Bôcher, Professor A. S. Chessin, Miss E. M. Coddington, Professor F. N. Cole, Dr. E. B. Cowley, Dr. F. J. Dohmen, Professor L. P. Eisenhart, Professor W. H. Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor G. H. Ling, Dr. E. N. Martin, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Professor D. E. Smith, Professor P. F. Smith, Dr. W. M. Strong, Mr. J. S. Thompson, Professor H. W. Tyler, Professor Oswald Veblen, Mr. H. E. Webb, Miss M. E. Wells, Professor H. S. White, Professor J. E. Wright.

The President of the Society, Professor Maxime Bôcher, occupied the chair, being relieved at the afternoon session by Vice-President Professor Edward Kasner. The Council announced the election of the following persons to membership in the Society: Miss W. B. Bauer, High School, Topeka, Kan.; Mr. W. W. Denton, University of Illinois; Mr. Meyer Gaba, University of Kansas; Professor W. H. Garrett, Baker University; Miss S. E. Graham, High School, Topeka, Kan.; Mr. G. F. Gundelfinger, Sheffield Scientific School; Professor W. A. Harshbarger, Washburn College; Mr. L. T. W. Hogrefe, Milwaukee, Wis.; Professor L. A. Howland, Wesleyan University; Mr. George Melcher, State Normal School, Springfield, Mo. Three applications for membership in the Society were received.

Professor H. S. White was reelected a member of the Editorial Committee of the Transactions for a term of three years beginning October 1, 1909.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On certain functional equations."

(2) Professor R. D. CARMICHAEL: "Note on some polynomials related to Legendre's coefficients."

(3) Professor W. H. JACKSON: "A theorem concerning simple continued fractions."

(4) Professor W. H. JACKSON: "Shadow rails."

(5) Professor L. P. EISENHART: "The twelve surfaces of Darboux and the transformation of Moutard."

(6) Professor W. F. OSGOOD: "On Cantor's theorem concerning the coefficients of a convergent trigonometric series, with generalizations."

(7) Mr. L. S. DEDERICK: "Certain singularities of transformations of two real variables."

(8) Professor PAUL SAUREL: "On Fredholm's equation."

(9) Professor J. E. WRIGHT: "On abelian functions of genus 3."

(10) Dr. J. C. MOREHEAD: "A simplification of Lagrange's method for the solution of numerical equations (second paper)."

(11) Professor A. S. CHESSIN: "On gyroscopic couples."

(12) Professor EDWARD KASNER: "The interpretation of differential equations in line coordinates."

(13) Professor E. D. ROE, Jr.: "On the extension of the exponential theorem."

Mr. Dederick's paper was presented to the Society through Professor Bouton. In the absence of the authors Mr. Dederick's paper was read by Professor Bôcher, and the papers of Professors Carmichael, Saurel, Chessin, Kasner, and Roe, and Dr. Morehead were read by title. Professor Saurel's paper appears in full in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael considers the equations

$$h(x+y)\,h(x-y) = [h(x)]^2 + [h(y)]^2 - c^2,$$

$$g(x+y)\,g(x-y) = [g(x)]^2 - [g(y)]^2.$$

He finds that the general solution of the first is

$$h(x) = \pm \tfrac{1}{2}c(e^{kx} + e^{-kx})$$

and shows that, except for a constant factor, several related functional equations have the same solution. The second of the given equations has two solutions

$$g(x) = \pm \frac{g'(0)}{2k}(e^{kx} - e^{-kx}), \quad \text{if} \quad k \neq 0,$$

$$g(x) = \pm g'(0)\ x, \quad \text{if} \quad k = 0,$$

where $g'(0)$ is an arbitrary constant.

2. In his second paper Professor Carmichael considers a class of functions

$$P_{na} = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^{2a} - 1)^n,$$

and shows that several properties of Legendre's polynomials are properties also of this generalized function. P_{n1} is the usual Legendre coefficient.

3. In his first paper Professor Jackson considers all those integral multiples of a given number for which the fractional part lies between given limits. It is shown that the integral multipliers satisfying this condition possess a periodic formation, one period within another, but that these periods are subject to breaks at calculable intervals.

4. In his second paper Professor Jackson applies the results of his first to calculate the position of the shadow rails observed when one set of rails is viewed through another. It is established that the spacing of the shadow bands depends only on the spacing of the two sets of rails and not on their breadths, and that when the position of the observer is altered the shadow bands move with magnified velocity in the same direction as that of the apparent motion of the more closely spaced set of rails relative to the other set.

5. The problem of the infinitesimal deformation of a surface S is equivalent to the determination of a surface S_1 corresponding to S with orthogonality of linear elements, or of an associate surface S_0 whose tangent plane is parallel to the corresponding tangent plane to S and whose asymptotic lines correspond to a conjugate system on S. In the fourth volume of his Leçons Darboux shows that nine other surfaces are associated with S, S_1, S_0 in such a way that any three successive surfaces of the closed cycle are in relations similar to those which hold between S, S_1, and S_0. Darboux is led to the result that the cycle is

closed after he has calculated the coordinates of the various surfaces. Professor Eisenhart, in the first part of his paper, is concerned with the reasons for this closure and with an investigation as to the possibility that all of the twelve surfaces be not distinct. In the second part of his paper, he considers the group of surfaces which figure in the infinitesimal deformations of a surface S and the three surfaces S', S'', S''' which arise from S by transformations of Moutard,* these surfaces being such that the pairs S, S'; S, S''; S', S'''; S'', S''' are the focal surfaces of W-congruences. In general there are sixteen distinct surfaces in the group. An exceptional case of importance is that for which the two surfaces associate to S are associate to one another.

6. Professor Osgood's paper contains a simple proof of the theorem that if a trigonometric series

$$\Sigma(a_n \cos nx + b_n \sin nx)$$

converges for all values of x in a certain interval, however short, its coefficients approach zero as their limit. The theorem is generalized to series of the form $\Sigma A_n f_n(x)$, where f is continuous in an interval (a, b) and in every subinterval (α, β) takes on values numerically greater than a certain positive constant l at some point of the interval, provided $n \geqq m$. If such a series converges in (a, b), then

$$\lim_{n=\infty} A_n = 0.$$

Further generalizations to multiple series

$$\Sigma f_{m,n}(x, y), \quad \Sigma f_{m,n,p}(x, y, z),$$

are given.

The paper will be offered for publication in the *Transactions*.

7. The object of Mr. Dederick's paper is to discuss general properties of those singularities of transformations of two real variables $u = \phi(x, y)$, $v = \psi(x, y)$ which are due not to discontinuities in the functions ϕ or ψ, or their partial derivatives, but to the vanishing of the Jacobian determinant $J = \phi_x \psi_y - \phi_y \psi_x$. The most important results are the derivation and use of a general formula for the transformation of the derivative of one variable with respect to the other, which applies with an unimportant exception to all singularities of

* Bianchi, Lezioni, vol. 2, p. 69.

this sort, and the classification of such singularities on the basis of this formula. The transformed second derivative is in general infinite except in the simplest case. Some geometric properties discussed are those of the critical slopes, the preservation of regularity of an arc, the criterion for the transformation of a curve into a point, and the order of contact of the curves $\phi = \text{const.}$ and $\psi = \text{const.}$ with each other and with $J = 0$. A reduction of the transformation to the normal form, $u_1 = x_1$, $v_1 = y_1^2$, is obtained for any neighborhood of an arc of $J = 0$ in which one of the expressions $\phi_x J_y - \phi_y J_x$ and $\psi_x J_y - \psi_y J_x$ remains different from zero, the reduction being effected by transformations of the two planes whose Jacobians nowhere vanish. The following theorem on implicit functions is derived: If $F(u, x)$ and all its partial derivatives of order less than n vanish at (u_0, x_0) and the partial derivatives of the nth order are continuous near (u_0, x_0) and not all zero at that point, then to every real simple root of the equation of the nth degree for formally determining du/dx, corresponds a solution $u = f(x)$ which has a continuous derivative near (u_0, x_0).

9. Professor Wright's paper is concerned with the determination of the differential equations satisfied by abelian θ functions of genus 3. The equations are obtained in covariantive form, and they lead to the generalization of Kummer's quartic surface. This turns out to be a three spread in space of six dimensions, of the 8th order in the hyperelliptic, of the 16th order in the non-hyperelliptic case.

10. In an earlier paper* Dr. Morehead showed how binomial synthetic division may be applied to the approximate expression of the incommensurable real roots of numerical equations in terms of simple continued fractions

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}.$$

This method is modified, in the present paper, so as to give the approximate expression of the roots in continued fractions of the form

$$a_1 + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \cdots + \frac{b_n}{a_n},$$

the b's being arbitrarily chosen.

* Read before the Chicago Section, April 9, 1909.

The similar process for the expression of the roots in ascending continued fractions

$$a_1 + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}$$

is a generalization of Horner's method, as the above reduces to the decimal fraction $a_1 a_2 a_3 \cdots a_n$ for $b_2 = b_3 = \cdots = b_n = 10$.

11. The well known property of rapidly revolving bodies to resist efforts tending to deflect their principal axes is due to the rise of couples among which that of the so-called turning forces produces most of the effect. This couple is generally assumed to be equal to $C\omega\phi' \sin\theta$ where C, ω, ϕ, and θ denote respectively the moment of inertia about the axis of spin, the angular velocity of deflection, the angular velocity of spin S, the angle of inclination between the axis of spin and the axis of deflection. In the present paper Dr. Chessin shows that the correct expression of the turning couple is given by the following values of its projections for the three eulerian axes:

$$C\omega(\phi' + \omega\cos\theta)\sin\theta, \quad (2A - C)\omega\theta'\cos\theta, \quad C\omega\theta'\sin\theta,$$

which reduce to the commonly used approximate value $C\omega\phi'\sin\theta$ when $\theta' = 0$ and ω is very small compared to ϕ'.

12. An oriented straight line may be defined most simply by means of its Hessian coordinates (u, v), the latter representing the length of perpendicular dropped from the pole and the former representing the angle which this perpendicular makes with the initial line. A differential equation of the form $F(dv/du, u, v) = 0$ then defines a single infinity of oriented curves. The interpretation of derivatives is very simple. Professor Kasner discusses in particular the systems of curves represented by linear and Riccati equations. These types may be regarded as natural extensions of systems of parallel curves. In constrast with the usual statements for cartesian coordinates (x, y), the results are here invariant under the displacement group.

13. Professor Roe's paper contains an elaboration of the theory of the exponential function.

F. N. COLE,
Secretary.

APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-fifth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, April 9–10, 1909. The extensive programme required four full half-day sessions. Professor G. A. Miller, Chairman of the Section presided except on Friday afternoon when Professor E. B. Van Vleck, Vice-President of the Society, occupied the chair. The attendance at the various sessions included sixty-two persons among whom were the following forty-six members of the Society:

Mr. R. P. Baker, Mr. W. H. Bates, Dr. G. D. Birkhoff, Professor G. A. Bliss, Mr. Thomas Buck, Mr. H. E. Buchanan, Professor D. F. Campbell, Professor H. E. Cobb, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor J. C. Fields, Professor W. B. Ford, Professor C. N. Haskins, Mr. T. H. Hildebrandt, Professor T. F. Holgate, Professor Kurt Laves, Dr. A. C. Lunn, Dr. E. B. Lytle, Professor Max Mason, Professor Malcolm McNeill, Mr. E. J. Miles, Dr. W. D. MacMillan, Professor G. A. Miller, Professor E. H. Moore, Dr. R. L. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Mrs. Anna J. Pell, Professor Alexander Pell, Professor C. A. Proctor, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor C. S. Slichter, Professor G. T. Sellew, Professor J. B. Shaw, Professor H. E. Slaught, Professor E. J. Townsend, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor A. E. Young, Professor J. W. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening forty members of the Society dined together in the café of the university commons and discussed informally various topics of interest, including the plans for the summer, 1909, meeting of the British association for the advancement of science to be held at Winnipeg, and the next International congress of mathematicians to be held in England in 1912. Professor J. C. Fields, of Toronto, extended a cordial invitation to the members to attend the former, and Professor E. H. Moore urged upon all the desirability of laying plans now to attend the latter.

Particular interest centered in the report of the committee appointed at the January, 1909, meeting to consider plans for improvement in the methods of making mathematical appointments in this country. In case a suitable plan had approved itself to the committee and to the Section, it was the intention to present this plan to the Council for its consideration. The committee, after careful deliberation, reached the conclusion that, while such improvement is manifestly desirable in many American colleges and universities, they were unable to recommend for consideration any plan for such improvement by action of the Society. The Section accepted the report and requested the Secretary to arrange for the publication of a plan which was proposed tentatively by Professor Wilczynski, chairman of the committee, and which he described in some detail. This will appear later in the BULLETIN.

The following papers were read at this meeting:

(1) Professor C. N. HASKINS: "On a class of discontinuous functions of two variables."

(2) Mr. A. E. WESTERN and Dr. J. C. MOREHEAD: "The Fermat number $2^{256}+1$."

(3) Dr. J. C. MOREHEAD: "A simplification of Lagrange's method for the solution of numerical equations."

(4) Mr. THOMAS BUCK: "Oscillations near Lagrange's equilateral triangle points in the problem of three bodies."

(5) Professor L. E. DICKSON: "On the equivalence of pairs of quadratic forms under rational transformation."

(6) Dr. ARTHUR RANUM: "The group of classes of quadratic integers with respect to a composite ideal as modulus."

(7) Professor A. E. YOUNG: "On surfaces having isotherm-conjugate lines of curvature."

(8) Professor W. B. FORD: "A set of criteria for the summability of divergent series."

(9) Professor W. B. FORD: "On the determination of the asymptotic developments of a given function."

(10) Mr. H. E. BUCHANAN: "A class of periodic orbits of three finite bodies."

(11) Professor G. A. MILLER: "Automorphisms of order two."

(12) Mr. R. P. BAKER: "The attack on the space and time concepts by Einstein and Minkowski."

(13) Professor E. W. DAVIS: "An imaginary conic."

(14) Dr. L. I. NEIKIRK: "Groups of rational integral transformations in a general field."

(15) Professor L. E. DICKSON: "A theory of invariants."

(16) Professor L. E. DICKSON: "Combinants."

(17) Mr. W. H. BATES: "Maschke's symbolic method applied to some questions in geometry of hyperspace."

(18) Professor L. D. AMES: "A simpler proof of Lie's theorem for ordinary differential equations."

(19) Dr. W. D. MACMILLAN: "Periodic orbits about an oblate spheroid."

(20) Professor H. L. RIETZ: "On the effect of types of correspondence on Bravais's coefficient of correlation."

(21) Professor F. R. MOULTON: "Oscillating satellites when the finite bodies describe elliptic orbits."

(22) Professor E. J. WILCZYNSKI: "Projective differential geometry of developables."

(23) Mr. ARTHUR PITCHER: "Complete elementary theory of certain properties of classes of functions."

(24) Mr. T. H. HILDEBRANDT: "Remarks on the general theory of point sets."

(25) Mrs. ANNA J. PELL: "Biorthogonal systems."

(26) Professor F. R. MOULTON and Dr. W. D. MACMILLAN: "On the solution of linear differential equations with periodic orbits."

Mr. Pitcher was introduced by Professor Moore. In the absence of the authors the papers of Professors Ames and Davis and Dr. Ranum were read by title. Abstracts of the papers follow below. The numbering corresponds to the titles in the above list.

1. The note of Professor Haskins calls attention to functions of the types

$$Z(x, y) = \frac{yu(x)}{y^2 + [u(x)]^2}, \quad \text{and} \quad Z(x, y) = \frac{ye^{-\frac{1}{[u(x)]^2}}}{y^2 + e^{-\frac{2}{[u(x)]^2}}},$$

where $\lim_{x=0} u(x) = 0$. The first of these functions has the property that

$$\lim_{x=0} Z(x, mv(x)) = 0 \quad \text{if} \quad \lim_{x=0} \frac{u(x)}{v(x)} = 0 \quad \text{or} \quad \lim_{x=0} \frac{v(x)}{u(x)} = 0,$$

and

$$\lim_{x=0} Z(x, mv(x)) = \frac{m\alpha}{m^2 + \alpha^2} \quad \text{if} \quad \lim_{x=0} \frac{u(x)}{v(x)} = \alpha \neq 0.$$

This gives rise to a set of functions which show the discontinuity in question along systems of curves having osculation with $y = 0$ of continually increasing order at the origin.

2. This paper describes the method of computation followed by Mr. Western and Dr. Morehead which resulted in showing that the Fermat number $F_8 = 2^{256} + 1$ fails to satisfy the criterion for Fermat primes,*

$$3^{2^{(2^n-n-2)}} \equiv \pm 2^r(2^{2^n-1} \pm 1) \mod F_n, \quad (F_n = 2^{2^n} + 1),$$

and is therefore composite. The result is especially interesting as completing a chain of five composite Fermat numbers, F_5, F_6, F_7, F_8, F_9, following the first five, and only known Fermat primes, 3, 5, 17, 257, 65537.

3. In this paper Dr. Morehead explains the application of binomial synthetic division to the approximate expression of an incommensurable real root of a numerical equation,

$$f_1(x) = c_0x^m + c_1x^{m-1} + \cdots + c_m = 0,$$

in terms of a continued fraction of the form

$$(a_1, a_2, a_3, \cdots a_n) = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}.$$

The quotients $a_1, a_2, a_3, \cdots, a_n$ are the integral parts of corresponding roots of the equations

$$\begin{aligned}
f_1(x_1) &= c_1x_1^m + c_1x_1^{m-1} + \cdots + c_m = 0,\\
f_2(x_2) &= c_0''x_2^m + c_1''x_2^{m-1} + \cdots + c_m'' = 0,\\
f_3(x_3) &= c_0'''x_3^m + c_1'''x_3^{m-1} + \cdots + c_m''' = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
f_n(x_n) &= c_0^{(n)}x_n^m + c_1^{(n)}x_n^{m-1} + \cdots + c_m^{(n)} = 0,
\end{aligned}$$

respectively, where each equation $f_{k+1}(x_{k+1}) = 0$ is connected with the preceding $f_k(x_k) = 0$ by the relation $x_k = a_k + 1/x_{k+1}$, or $x_{k+1} = 1/(x_k - a_k)$. From the latter relation it is easily seen that the coefficients of $f_{k+1}(x_{k+1})$ may be obtained from

* A modification, due to A. E. Western, of the Pepin criterion for the base 3: $3^{2^{(2^n-1)}} \equiv -1 \mod F_n$.

those of $f_k(x_k)$ if we diminish the roots of $f_k(x_k)$ by a_k, using the binomial synthetic division process as in Horner's method, and reverse the order of the coefficients so obtained.

Approximations for the roots of numerical equations may be obtained by this means more easily and more rapidly, in general, than by Horner's method. Thus for example, without using more than six figures in any of the coefficients, we obtain for the root lying between 1 and 2 of the equation,

$$2x^3 - x^2 + 2x - 4 = 0,$$

the approximation (1, 6, 1, 3, 1, 1, 3, 2, 3, 1, 4, 1, 1, 3, 1, 1, 5, 2, 1, 2, 2, 3) = 82994039/72327612, which is correct to 2 in the sixteenth decimal place.

4. In the first part of this paper Mr. Buck considers certain periodic orbits of an infinitesimal body attracted by two finite bodies revolving in circles about their center of mass. Convenient parameters are introduced and by a discussion of the linear terms the generating solutions are found. All the terms of the differential equations are then considered and the question of the existence of the continuations of the generating solutions is discussed.

In the second part of the paper certain periodic orbits of three finite bodies near Lagrange's equilateral triangle points are considered. Parameters are introduced as before and the linear terms discussed. The discussion when the higher terms of the differential equations are considered has not yet been completed.

5. The first paper by Professor Dickson considers the equivalence of two pairs of quadratic forms ϕ, ψ and ϕ', ψ' under linear transformation in a field F. As in the algebraic theory, the elementary divisors of $\phi - \lambda\psi$ must be the same as those of $\phi' - \lambda\psi'$. But this condition is not in general sufficient for equivalence in F. We extend the field F to a field F' by adjoining the roots λ_i of $|\phi - \lambda\psi| = 0$. If ϕ_1, ψ_1 is a canonical type of ϕ, ψ within F', each transformation effecting the reduction is the product of one with conjugate variables by an automorph of ϕ_1, ψ_1. A discussion of the transformations of ϕ_1, ψ_1 into ϕ_1', ψ_1' leads to conditions for their equivalence within F'. The conjugacy of the variables enables us to decide upon the equivalence of the initial pairs within F. Use

is made of the work of Kronecker and a modified form of that of Weierstrass. Both in the singular and in the non-singular cases, simple necessary and sufficient conditions are obtained for the equivalence of two pairs of forms under linear transformation in the given field. This paper has been offered for publication in the *Transactions*.

6. In the ordinary theory of rational numbers the group of classes of congruent integers with respect to a composite modulus is well known and is called by Weber is his Algebra the most important example of a finite abelian group. In the more general theory of numbers in an algebraic field the corresponding group of classes of integers with respect to a composite ideal does not seem to have been studied heretofore. In this paper a beginning is made by Dr. Ranum. For the special case of a quadratic field $(\sqrt{m})$ he determines the structure of the group, and finds a system of independent generators. If the modulus is a power of a prime ideal p, the latter being a factor of the principal ideal (p), the group is one of nine distinct types, depending on the value of m, the grade of p, and the value of the rational prime p ($p=2$, $p=3$, or $p>3$). If the modulus contains as factors two or more distinct prime ideals, the group is a direct product of two or more groups of these nine types. The cases in which the group is cyclic, and therefore primitive roots exist, are enumerated.

7. In a paper published in the January issue of the *Transactions*, Professor Young discussed a class of surfaces characterized by the two properties that their lines of curvature form a network of infinitesimal squares, and their asymptotic lines form a network of rhombuses. In the present paper he discusses the more general class, characterized by the latter of these two properties. The general problem of determining these surfaces is reduced to the simultaneous solution of two partial differential equations of the second order which involve two functions. Particular solutions are found and the corresponding surfaces determined. It is shown that these surfaces compose the class which have isotherm-conjugate lines of curvature, using this term in the sense defined by Bianchi.

8. Professor Ford's first paper appears in full in the present number of the BULLETIN.

9. The object of Professor Ford's second paper is to prove, and show the range of applicability of, a general formula employed extensively of late years by Barnes and made basal by him in a series of memoirs upon the determination of the asymptotic developments of a given function in the neighborhood of the point ∞. The formula in question is *

$$\sum_{n=1}^{m-1}\phi(n) \sim Z + \int \phi(m)dm - \tfrac{1}{2}\phi(m) + \sum_{n=0}^{n=\infty} \frac{(-1)^n B_{n+1}}{(2n+2)!}\frac{d^{2n+1}\phi(m)}{dm^{2n+1}},$$

where Z is a certain constant and B_n the nth Bernoulli number. It is shown that the class of functions $\phi(n)$ to which the formula applies is well defined in that any such function must be developable in a prescribed way about the point $n = \infty$.

10. At the January meeting of the Section, Mr. Buchanan read a preliminary report on a certain class of periodic orbits of three finite bodies. He showed how to expand the right members of the differential equations and discussed the linear terms. In the present paper it is proposed to introduce certain convenient parameters, prove the existence of five different types of orbits, show how to construct these orbits, and state certain theorems concerning their geometrical properties.

11. The necessary and sufficient condition that an automorphism is of order 2 is that every commutator arising from the automorphism corresponds to its inverse under the automorphism. Hence it results that in every automorphism of order 2 some operator besides identity must correspond to its inverse. The independent generators of an abelian group of odd order may be so chosen that in any given automorphism of order 2 each of these generators corresponds either to itself or to its inverse. If all the commutators (besides identity) of a group are of order 2, every complete set of conjugate operators of order 2 generates an abelian group of type $(1, 1, 1, \cdots)$, and such a group involves an abelian subgroup which is the direct product of Sylow subgroups of all odd orders. Every group in which the orders of all the commutators are powers of the same prime number is solvable, and such a group involves only one Sylow subgroup whose order is a power of this prime.

* *Philosophical Transactions*, vol. 199 A (1902), p 444.

12. The validity of the time and space concepts has been challenged by Einstein * and Minkowski.† Mr. Baker's paper is a critical examination of their theory.

Einstein's definition of time is insufficient; *fluit* must be added. Velocities in (ξ, η, ζ, τ) are non-associative in Einstein's system. Minkowski avoids this by his definition of "Eigenzeit."

The usual treatment of relativity depends on a transformation and its inverse, and on the ordinary dualism. T moves the object and T^{-1} the observer. If the dualism is not abandoned and O represents the ordinary transformation for uniform motion, the selection of OT^{-1} is only possible on a strict definition of *object* as a totality C_3 of light sources which act on moving according to Einstein's hypothesis, that is, are each a totality of pulses (C_1), the pulse being invariant under T and Einstein's source remaining a C_1 of pulses under O. Cross selection in the multiple infinities must be barred and each time t must be associated with its place.

With these restrictions the system becomes a single sense philosophy, the "sense organ" accomplishing the accurate timing of light signals by "synchronized clocks."

The selection of TT^{-1} as the pair is without special significance for relativity, though T of course has its very important significance for the problem of relative motion.

Since O does not preserve the equations of electrodynamics it is probable that Einstein's success with OT^{-1} is due to abstraction.

At any rate "absolut ruhender Raum" as applied to (x, y, z, t) is erroneous. When the transformation is obtained it turns out that there is no analytical distinction between this system and (ξ, η, ζ, τ) beyond the sign of v. Minkowski postulates a world. He asks for (1) continuity in four dimensions (x, y, z, t), (2) each point to be a substantial point, with assigned mass, (3) $ds/dt < V$, (4) a "Geschenk von oben," the contraction of moving bodies in the direction of their motion according to Einstein's law.

In the actual work "each" in (2) is defined by a continuous function, thus avoiding an obvious difficulty with the point set theory.

The contraction in (4) is not an affine transformation, the

* *Annalen der Physik*, vol. 17 (1905).

† *Physikalische Zeitschrift*, Feb., 1909. *Göttinger Nachrichten*, Dec., 1908.

factor for the inverse transformation being the same as for the direct, and not the reciprocal of it. An ordinary hydrodynamic universe in (x, y, z, t) with an affine contraction for (4) has for its only rotating molecules those with fixed axis, size, and angular velocity, posited as eternal.

The erroneous interpretation of the contraction is however a matter of indifference; satisfaction of the equations is all that is necessary.

13. In this paper Professor Davis points out that the conic $\bar{x}^2 + \bar{y}^2 e^{2i\phi} = 1$ can be gotten by revolving the intersection of the cylinder $x^2 + y^2 = 1$ with $z = iy$ about the x-axis through an angle ϕ, and then projecting the intersection upon the xy-plane. The distribution of the imaginary elements is indicated by a diagram.

14. In this paper Dr. Neikirk considers the totality of transformations of the type

$$T_{m,i} \equiv [x : \phi_{m,i}(x)],$$

where

$$\phi_{m,i}(x) = \sum_{j=0}^{j=m} \alpha_{j,i} x^{m-j} \qquad (\alpha_{o,i} \neq 0),$$

and the coefficients range independently over the elements or marks of a general field F.

The successive application of two of these transformations is a definite third one in the set. Symbolically $T_{m,i} T_{m',i'} = T_{mm',i''}$. They possess the group property. They also obey the associative law, if x is unrestricted. If x is restricted in range to a certain set, then the result of the action of $T_{m,i}$ on x, *i. e.*, on $\phi_{m,i}(x)$ must be in this same set for associativity.

Not all the transformations in this set have an inverse in the set. If $T_{m,i}$ $(m > 1)$ has $T_{m',i'}$ as an inverse, then x is restricted in range to the roots of an algebraic equation with coefficients in F, and $T_{m,i}$ gives a substitution on the roots of this equation. These transformations on the roots of an equation are a two-fold generalization of the substitution quantics given by Hermite, Dickson, and others. Several theorems are proved and several examples are given of groups generated by these transformations.

15. In the second paper by Professor Dickson, the invariants of a system of forms are investigated from the standpoint

of the classes of forms under the group of transformations of determinant unity. Whereas the discussion in the April number of the *Transactions* was limited to the case of a finite field, the present paper deals with an arbitrary field. For the case of the field of all real and complex numbers, the invariants considered are the rational integral invariants in the ordinary algebraic theory, as well as other single-valued invariants such as the rank of a quadratic form. Since all the single-valued invariants appear from a common standpoint, the present theory presents a synthesis of the special theories. The paper has been offered to the *American Journal of Mathematics.*

16. In the third paper by Professor Dickson, s-fold families $\Sigma x_i q_i$ of forms q_i on m variables ξ_j are separated into classes under the group Γ_1 of m-ary linear transformations on the ξ_j and the group G_1 of s-ary linear transformations on the x_i, each type of transformations having determinant unity. A function of the coefficients of the q_i is invariant under Γ_1, G_1 if and only if it takes the same value for all forms $\Sigma x_i q_i$ in any class. For a finite field the number of the resulting linearly independent invariants equals the number of classes. By an extension of a method given elsewhere,* it is determined which of these invariants are combinants; the number of linearly independent combinants is shown to equal the number of classes under Γ_1, G_1. For the practical determination of combinants various methods are developed. The general theory is applied to determine a fundamental system of five (ten) combinants of two binary (ternary) quadratic forms in the field of order p^n, $p > 2$. That the combinants actually form a fundamental system follows from a general theorem for finite and infinite fields (cf. preceding abstract). The paper has been offered to the *Quarterly Journal of Mathematics.*

17. In this paper, Mr. Bates uses Maschke's symbolic method to derive some of the formulas relating to angles between curves, isometric systems, and lines of curvature in hyperspace.

18. Professor Ames's paper appeared in full in the May BULLETIN.

19. The motion of a particle about an oblate spheroid is not, in general, periodic, but by virtue of the existence of one area

* *Transactions*, April, 1909, § i.

integral it is possible to reduce the problem from three dimensions to two. In the particular case of motion in the equatorial plane Dr. Macmillan shows that, within a certain range of variation of the parameters, the expression for the radius vector is always periodic. The expression for the rotation of the line of apsides has an interesting application to the motion of Jupiter's fifth satellite.

When the motion is not in the equatorial plane it is possible always to consider the motion in a revolving meridian plane. An orbit is considered to be periodic when its motion in this plane is periodic. It is found that the inclination can be taken arbitrarily and the initial velocity so determined that the orbit in the revolving plane is closed after the first revolution. These are the simplest types of orbits. Let us say that they have the period 2π. On varying the initial distance slightly it is found that a new period $2\pi/\lambda$ is introduced. If λ is rational, it is possible to determine the change in the initial velocity in terms of the initial displacement so that the orbit is reentrant after many revolutions. The integration of non-homogeneous linear differential equations with periodic coefficients is involved in every step. These last solutions include five constants of integration out of the six necessary for a complete solution, with the condition that λ must be rational.

20. In this paper, Professor Rietz considers the effect of some different types of correspondence on results obtained by Bravais's formula for correlation. When a one to one correspondence exists for variates of the subject and relative classes, in treating the correlation of two attributes by statistical methods, the formula $r = \sum xy/n\sigma_x\sigma_y$ is uniquely defined. But a more general type of correspondence exists in some practical problems of statistics. A μ to ν correspondence occurs in which μ and ν are constant, but the μ or ν variates are not identical. In this case, it is a somewhat arbitrary matter as to which one of three or more distinct methods is used to obtain a one to one correspondence for application of the formula. It turns out that, with three different methods, which give significant measures of correlation, the values of r are in two of them different; and their numerical order is determined. In particular, if $\mu = 1$, although the correlation coefficients differ, the regression coefficients of the relative on the subject class are equal for two of the methods of correspondence; and this value

of the regression coefficient is the most probable value for the third method if the frequency distributions are symmetrical.

21. It is possible to project any three bodies so that they shall always be collinear and describe conic sections. Suppose one of the three is infinitesimal and that the orbits are ellipses. Then Professor Moulton shows that it is possible to displace the infinitesimal body from its position of equilibrium in an infinity of different ways, both in the plane of motion of the finite bodies and in three dimensions, so that its motion shall be periodic. When the finite bodies describe circles the orbit of the infinitesimal body, referred to rotating axes, re-enters after one revolution, but in the present case it re-enters only after many revolutions. The orbits depend upon a certain parameter and they are periodic for an infinite number of values of it, but these values do not constitute the continuous point set, for they must be such that certain commensurability conditions are satisfied.

22. In this paper Professor Wilczynski indicates a method for the development of a projective differential geometry of developable surfaces. These surfaces cannot be treated by the general methods of his previous papers, based upon the consideration of two linear homogeneous partial differential equations of the second order, because for a developable the corresponding system would become involutory, so that the most general integral would not be a projective transformation of any one. The difficulty is solved by adjoining an independent partial differential equation of the third order, to the involutory system of two equations of the second order. Mr. W. W. Denton is at present developing the details of this theory. It should be mentioned, however, that the dual theory, in which a developable is considered as an envelope of ∞^1 planes, is already contained in Professor Wilczynski's previous investigations.

23. The classes of functions considered by Mr. Pitcher are classes $\mathfrak{M}$ on a general class $\mathfrak{P}$ to $\mathfrak{A}$, where $\mathfrak{A}$ is the class of all real numbers. Certain ten properties of classes of functions have been found by Professor Moore to be very important in his work in general analysis. Four of these properties have to do with systems $(\mathfrak{A}, \mathfrak{P}, \mathfrak{M})$, while the others have to do with systems $(\mathfrak{A}, \mathfrak{P}, \Delta, \mathfrak{M})$ where Δ is a so-called development of $\mathfrak{P}$.

The objects of the present study are: (1) To investigate the elementary relations of dependence and independence of the ten properties mentioned above, (2) to give characterizations in terms of these properties of certain classes of elements $\mathfrak{P}$ and of functions $\mathfrak{M}$. So far the study has been confined to seven of the ten properties. These seven are not completely independent. Various characterizations for $\mathfrak{P}$ singular, $\mathfrak{P}$ dual, $\mathfrak{P}$ finite, and for certain classes of functions have been found.

24. A critical study of Fréchet's thesis * has revealed the fact that in order to obtain a theory of continuous functions, in particular the theory obtained by him, it is not necessary to condition the notion of limit further than to suppose it to be a relation between sequences of elements and single elements. In order to obtain a more general setting for the second part of his study, which concerns a distance function of pairs of elements, *voisinage*, a relation between pairs of elements and integers has been introduced. By the use of suitable conditions on this relation, Mr. Hildebrandt is able to obtain the entire theory of Fréchet and also the extension made by Hahn.† Other questions of independence are being investigated.

25. In Mrs. Pell's paper, necessary and sufficient conditions are found for the existence of the adjoint system of a given system of functions. The system is shown to be denumerable if the adjoint system exists. Conditions for the convergence of

$$\sum_{i=1}^{\infty} \int_a^b f(s) X_i(s) ds \int_a^b g(s) Y_i(s) ds$$

are deduced and it is shown that this sum is equal to

$$\int_a^b f(s) g(s) ds$$

when the given system is complete.

26. Special cases of the differential equations treated in this paper have arisen in the investigations of Professor Moulton and Dr. MacMillan on periodic orbits, and the methods of solving them were first developed in connection with these

"Sur quelques points du calcul fonctionnel," *Rend. del Cir. di Palermo*, vol. 22 (1906).

† Bemerkungen zu den Untersuchungen des Herrn Fréchet, *Monatshefte f. Math. u. Phys.*, vol. 19 (1908).

practical problems. This discussion is, in fact, reduced to that of periodic solutions of an associated set of differential equations. After giving a new derivation of the properties of the solutions in the general case, the class of special equations

$$(1) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \left[a_{ij} + \sum_{\kappa=1}^{\infty} \theta_{ij}^{(\kappa)}(t)\mu^{\kappa} \right] x_j \qquad (i = 1, \cdots, n)$$

is treated. The θ_{ij} are periodic in t with the period 2π. These equations include those which generally arise in practical problems.

When $\mu = 0$ let the roots of the characteristic equation of the resulting differential equations be $\alpha_1^{(0)}, \cdots, \alpha_n^{(0)}$. It is proved that if $\alpha_1^{(0)}, \cdots, \alpha_n^{(0)}$ are distinct and if no two of them differ by an imaginary integer, then the general solution of (1) is

$$(2) \qquad x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t} y_{ij}(t) \qquad (i = 1, \cdots, n),$$

where the A_j are arbitrary constants, the y_{ij} are periodic with the period 2π, and

$$(3) \qquad \alpha_j = \alpha_j^{(0)} + \sum_{\kappa=1}^{\infty} \alpha_j^{(\kappa)}\mu^{\kappa}, \quad y_{ij} = \sum_{\kappa=0}^{\infty} y_{ij}^{(\kappa)}\mu^{\kappa}.$$

If two or more $\alpha_j^{(0)}$ differ by imaginary integers the solutions have *in general* the form (3), but there are interesting peculiarities both in the proof of their existence and in their construction.

If p of the $\alpha_j^{(0)}$ are equal, then in general the coefficients of $e^{\alpha_j t} y_{ij}(t)$ in (2) are polynomials in t and

$$(4) \qquad \alpha_j = \alpha_j^{(0)} + \sum_{\kappa=1}^{\infty} \alpha_j^{(\kappa)}\mu^{\kappa/p}, \quad y_{ij} = \sum_{\kappa=0}^{\infty} y_{ij}^{(\kappa)}\mu^{\kappa/p}.$$

In all the special cases which can arise convenient methods for constructing the solutions follow from the mode of treatment.

H. E. SLAUGHT,
Secretary of the Section.

A SET OF CRITERIA FOR THE SUMMABILITY OF DIVERGENT SERIES.

BY PROFESSOR WALTER B. FORD.

(Read before the Chicago Section of the American Mathematical Society, April 9, 1909.)

Introduction.

1. If the series

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots \tag{1}$$

be divergent but of indeterminacy r in Cesàro's sense, its sum is defined by the relation

$$s = \lim_{n=\infty} \frac{f(n, r)s_0 + f(n-1, r)s_1 + \cdots f(1, r)s_{n-1} + f(0, r)s_n}{f(n, r) + f(n-1, r) + \cdots f(1, r) + f(0, r)}, \tag{2}$$

where

$$f(n, r) = r(r+1)\cdots(r+n-1)/n!\ (r \geqq 1);\ s_n = u_0 + u_1 + \cdots + u_n$$

and where r is taken as the smallest integer for which the indicated limit exists. Since it is a necessary condition for indeterminacy r that

$$\lim_{n=\infty} \left(\frac{u_n}{n^r}\right) = 0 *$$

it follows that but few series (1) are summable by (2) for any value of r however large, and hence that summability in Cesàro's sense is of relatively rare occurrence. It is therefore proposed in the present paper to show how an extended set of criteria, of which (2) affords a special type, may be constructed so as to provide an ascending scale of sharpness for the testing of summability. We shall show that such criteria may be constructed so as to bear a close analogy to the familiar logarithmic criteria for testing the convergence or divergence of a given series.

Formulation of Problem and Theorem.

2. In order to state the problem in more accurate terms we recall that the essential characteristic of the sum formula (2),

*Cf. Bromwich, Infinite Series, p. 318 (1908).

or of any other general sum formula, is that it shall give $\lim_{n=\infty} s_n$ whenever the series (1) is convergent. Also we note that by increasing r in (2) we increase the order of infinitude of its denominator when $n = \infty$ and thereby increase the range of applicability of the formula. The following problem, which is the one we shall undertake, is therefore suggested: To determine an infinite sequence of functions

$$(3) \qquad f_0(n, r),\ f_1(n, r),\ f_2(n, r),\ \cdots,\ f_p(n, r),\ \cdots$$

each dependent upon the variable n and the parameter r and having the following characteristics:

$$(a) \qquad \lim_{n=\infty} \frac{\sum_{m=0}^{n} f_p(n-m,\ r)s_m}{\sum_{m=0}^{n} f_p(n-m,\ r)} = \lim_{n=\infty} s_n$$

whenever the series (1) is convergent;

$$(b) \qquad f_0(n, r) = \frac{r(r+1)\cdots(r+n-1)}{n!} \quad \text{(case of Cesàro);}$$

$$(c) \quad \lim_{n=\infty} \frac{\sum_{m=0}^{n} f_p(m, r)}{\sum_{m=0}^{n} f_p(m, r+1)} = 0, \qquad (d) \quad \lim_{n=\infty} \frac{\sum_{m=0}^{n} f_p(m, r)}{\sum_{m=0}^{n} f_{p+1}(m, 1)} = 0.$$

Any function of the sequence (3) will thus furnish an infinite subset of criteria having an increasing range of applicability with increasing r [see (c)], while in the case of a series (1) for which a given function of the sequence gives no meaning to the first member of (a) for any value whatever of r, we may at once obtain more powerful criteria by increasing p [see (d)]. Moreover, by virtue of (b) the extended set of criteria thus obtained will form a generalization of Cesàro's original formula (2).

In this connection, we shall now establish the following

THEOREM. *As a function $f_p(n, r)$ satisfying (a), (b), (c), (d) and therefore furnishing a set of criteria of summability for divergent series we may take*

$$f_p(n, r) = \frac{r(r+1)}{r + \log_p n} \prod_{n=2}^{n} \frac{r + \log_p n}{\log_p n},$$

where $\log_0 n = n$ *and* $\log_p n = \log\ \log \cdots \log\ n$ *(*p *times)*; $p = 1, 2, 3, \cdots$.

3. *Proof.* We shall first establish the following lemma: Let $c_0, c_1, c_2, \cdots, c_n$ and $C_0, C_1, C_2, \cdots, C_n$ be two sequences of quantities of which the elements of the latter are real and positive. If $\lim_{n=\infty} c_n = g$ and

$$(4) \qquad \lim_{n=\infty} \frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0} = 0, \quad m = \textit{fixed integer} \geqq 0,$$

then

$$(5) \qquad \lim_{n=\infty} \frac{C_n c_0 + C_{n-1} c_1 + \cdots + C_0 c_n}{C_n + C_{n-1} + \cdots + C_0} = g.*$$

In fact we shall then have $c_n = g + \epsilon_n$; $\lim_{n=\infty} \epsilon_n = 0$, so that we may write

$$\frac{C_n c_0 + C_{n-1} c_1 + \cdots + C_0 c_n}{C_n + C_{n-1} + \cdots + C_0} - g = X(n) + Y(n),$$

where

$$X(n) = \frac{C_n c_0 + C_{n-1} c_1 + \cdots + C_{n-m} c_m - g(C_n + C_{n-1} + \cdots + C_{n-m})}{C_n + C_{n-1} + \cdots + C_0},$$

$$Y(n) = \frac{\epsilon_{m+1} C_{n-m-1} + \epsilon_{m+2} C_{n-m-2} + \cdots + \epsilon_n C_0}{C_n + C_{n-1} + \cdots + C_0}.$$

But if C represent the largest of the quantities $|c_0|, |c_1|, \cdots, |c_m|$, we shall have

$$|X(n)| \leqq (C + g) \frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0}$$

and hence $\lim_{n=\infty} X(n) = 0$ by (4).

Again, corresponding to an arbitrarily small positive quantity ϵ we shall have

$$|Y(n)| \leqq \epsilon \frac{C_{n-m-1} + C_{n-m-2} + \cdots + C_0}{C_n + C_{n-1} + \cdots + C_0}$$

$$= \epsilon \left(1 - \frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0}\right), \quad m > m_\epsilon = \textit{constant},$$

and hence $\lim_{n=\infty} Y(n) = 0$. Thus the lemma becomes established.

* This is also true when each of the quantities $C_0, C_1, \cdots, C_n$ is a function of n, other conditions remaining the same, and the proof which follows will be found to apply equally to such cases.

In particular, let us now take

$$C_n = \frac{r(r+1)\cdots(r+n-1)}{n!}, \quad r = \textit{integer} \geqq 1,$$

and place $c_n = s_n$, where s_n represents the sum of the first $n+1$ terms of (1). Then condition (4) is satisfied and (5) is at once seen to give the sum formula (2) of Cesàro. More generally, if V_n be any function of n such that the difference $V_n - V_{n-1}$ never decreases as n increases (n sufficiently large), while at the same time

$$\lim_{n=\infty} \frac{V_n - V_{n-1}}{V_n} = 0, \tag{6}$$

we may obtain in like manner a sum formula for the series (1) by taking $C_n = V_n - V_{n-1}$. In fact, we shall then have

$$\frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0} \leqq (m+1)\frac{V_n - V_{n-1}}{V_n},$$

so that condition (4) becomes satisfied, with which (5) yields the formula mentioned. In the special case leading to formula (2) it will be observed that

$$V_n = \frac{r(r+1)\cdots(r+n)}{n!}; \quad V_n - V_{n-1} = \frac{r(r+1)\cdots(r+n-1)}{n!}, \tag{7}$$

so that

$$\frac{V_n - V_{n-1}}{V_n} = \frac{r}{r+n}.$$

We turn our attention therefore to the function V_n defined by a general equation of the type

$$\frac{V_n - V_{n-1}}{V_n} A_n; \quad \lim_{n=\infty} A_n = 0. \tag{8}$$

A solution of this equation which reduces to (7) when

$$A_n = \frac{r}{r+n}$$

(case of Cesàro) is

$$V_n = \prod_{n=1}^{n}\left(\frac{1}{1-A_n}\right). \tag{9}$$

Let us now consider the function V_n obtained by taking

$$A_1 = \frac{1}{r+1}, \quad A_n = \frac{r}{r+\log n} \qquad (n \geqq 2).$$

This function will necessarily satisfy (6) because of (8) and will be such that $V_n - V_{n-1}$ is an ever increasing function of n (n sufficiently large) since the result of increasing n by 1 is to multiply $V_n - V_{n-1}$ by

$$\frac{A_{n+1}}{A_n} \cdot \frac{V_{n+1}}{V_n} = \frac{r+\log n}{\log(n+1)} > \frac{\log(n+r)}{\log(n+1)} \geqq 1.$$

Thus we reach the sum formulas

$$(10) \quad s = \lim_{n=\infty} \frac{f_p(n, r)s_0 + f_p(n-1, r)s_1 + \cdots + f_p(0, r)s_n}{f_p(n, r) + f_p(n-1, r) + \cdots + f_p(0, r)},$$

where

$$f_p(n, r) = V_n - V_{n-1} = \frac{r(r+1)}{r+\log_p n} \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n} \quad (p = 0, 1)$$

with $\log_0 n = n$ (case of Cesàro) and $\log_1 n = \log n$.

More generally, by taking

$$A_1 = \frac{r}{r+1}, \quad A_n = \frac{r}{r+\log_p n},$$

where $\log_0 n = n$ and $\log_p n = \log \log \cdots \log n$ (p times) we are led to a function $f_p(n, r)$ defined by the formula

$$(11) \quad f_p(n, r) = \frac{r(r+1)}{r+\log_p n} \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n} \quad (p = 0, 1, 2, 3, \cdots),$$

which function satisfies conditions (a) and (b) of § 2.

In order to complete the proof of the theorem it remains therefore but to show that the function $f_p(n, r)$ defined by (11) satisfies (c) and (d).

Recalling that

$$\sum_{n=0}^{n} f_p(m, r) = \prod_{n=1}^{n} \left(\frac{1}{1-A_n} \right) = (r+1) \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n},$$

we have

$$(12)\quad \frac{\sum_{m=0}^{n} f_p(m, r)}{\sum_{m=0}^{n} f_p(m, r+1)} = \frac{r+1}{r+2} \prod_{n=2}^{n} \frac{r+\log_p n}{r+1+\log_p n},$$

$$= \frac{r+1}{r+2} \prod_{n=2}^{n} \left(1 - \frac{1}{r+1+\log_p n}\right),$$

and since the series

$$\sum_{n=2}^{\infty} \frac{1}{r+1+\log_p n}$$

is divergent, it follows from the well-known tests for convergence and divergence of infinite products that the last member of (12) approaches the limit 0 when $n = \infty$, with which condition (c) becomes satisfied.

Again we have

$$\frac{\sum_{m=0}^{n} f_p(m, r)}{\sum_{m=0}^{n} f_{p+1}(m, 1)} = \prod_{n=2}^{n} \frac{(r+\log_p n)\log_{p+1} n}{(1+\log_{p+1} n)\log_p n} = \prod_{n=2}^{n} (1-g_n),$$

where

$$g_n = \frac{\dfrac{1}{1+\log_{p+1} n} - \dfrac{r}{r+\log_p n}}{1 - \dfrac{r}{r+\log_p n}}.$$

For all values of n we thus have

$$g_n > \frac{\log_p n \left(1 - \dfrac{r \log_{p+1} n}{\log_p n}\right)}{(r+\log_p n)(1+\log_{p+1} n)} > \frac{c}{\log_{p+1} n};\ c = \textit{constant}$$

and hence the series

$$\sum_{n=2}^{\infty} g_n$$

is divergent, with which condition (d) becomes verified, thus completing the proof of the theorem.

ON FREDHOLM'S EQUATION.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 24, 1909.)

THE formulas which Fredholm has given* for the solution of the integral equation

$$\phi(x) + \lambda \int_0^1 \kappa(x, s)\phi(s)\,ds = \psi(x), \tag{1}$$

in which λ is a given constant, $\kappa(x, y)$ and $\psi(x)$ are given functions, while $\phi(x)$ is to be determined, are well known. Fredholm remarks that the theory of this equation may be considered as a limiting case of the theory of a set of linear algebraic equations, but in his published demonstration he makes no use of this remark and contents himself with a verification of his formulas. Hilbert† and Plemelj,‡ on the other hand, have obtained Fredholm's results by considering an infinite set of linear equations. More recently, Goursat § and Lebesgue || have shown that for a kernel $\kappa(x, y)$ having the form

$$\kappa(x, y) = X_1(x)Y_1(y) + X_2(x)Y_2(y) + \cdots + X_n(x)Y_n(y),$$

one can easily obtain Fredholm's formulas. In the present note, I should like to show how the consideration of equation (1) leads directly to Fredholm's expressions.

If we set

$$\phi(x) = \psi(x) - \rho(x), \tag{2}$$

* "Sur une nouvelle méthode pour la résolution du problème de Dirichlet," *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar*, Stockholm, vol. 57, p. 39 (1900). "Sur une classe de transformations rationnelles," *Comptes rendus*, vol. 134, p. 219 (1902). "Sur une classe d'équations fonctionnelles," *Acta Mathematica*, vol. 27, p. 365 (1903).

† "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Göttinger Nachrichten*, p. 49 (1904).

‡ "Zur Theorie der Fredholmschen Funktionalgleichung," *Monatshefte für Mathematik und Physik*, vol. 15, p. 93 (1904).

§ "Sur un cas élémentaire de l'équation de Fredholm," *Bulletin de la Société mathématique de France*, vol. 35, p. 163 (1907).

|| "Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholm," *Bulletin de la Société mathématique de France*, vol. 36, p. 3 (1908).

equation (1) becomes

$$\rho(x) + \lambda \int_0^1 \kappa(x, s)\rho(s)\,ds = \lambda \int_0^1 \kappa(x, s)\psi(s)\,ds. \tag{3}$$

Before attempting to solve this equation in all its generality let us first examine the special case in which $\psi(x)$ is identically equal to zero. The resulting equation

$$\rho(x) + \lambda \int_0^1 \kappa(x, s)\rho(s)ds = 0 \tag{4}$$

is obviously satisfied by taking $\rho(x)$ identically equal to zero. But the analogy that exists between this equation and a set of homogeneous linear algebraic equations makes it natural to attempt to determine the expression whose vanishing is necessary in order that the equation may admit of a solution not identically equal to zero.

From equation (4) we get

$$\rho(x) = -\lambda \int_0^1 \kappa(x, s_1)\rho(s_1)ds_1. \tag{5}$$

If we add to each side of this equation the expression

$$\lambda\rho(x)\int_0^1 \kappa(s_1, s_1)ds_1,$$

the integrand on the right will take the form of a determinant and we get

$$\rho(x)\left[1 + \lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_1) \\ \rho(s_1) & \kappa(s_1, s_1) \end{vmatrix} ds_1 \tag{6}$$

$$= \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_2) \\ \rho(s_2) & \kappa(s_2, s_2) \end{vmatrix} ds_2. \tag{7}$$

In the second member of equation (7) let us replace $\rho(x)$, $\rho(s_2)$ by the values given by equation (5); we get

$$\rho(x)\left[1 + \lambda\int_0^1 \kappa(s_1, s_1)ds_1\right]$$

$$= -\lambda^2\int_0^1\int_0^1 \rho(s_1)\begin{vmatrix} \kappa(x, s_1) & \kappa(x, s_2) \\ \kappa(s_2, s_1) & \kappa(s_2, s_2) \end{vmatrix} ds_1 ds_2. \tag{8}$$

An interchange of the subscripts 1, 2 in the expression on the right of this equation will not change its value; we may therefore write

$$\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right]=\frac{\lambda^2}{2}\int_0^1\int_0^1\left[-\rho(s_1)\begin{vmatrix}\kappa(x, s_1) & \kappa(x, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix}\right.$$

(9)

$$\left.-\rho(s_2)\begin{vmatrix}\kappa(x, s_2) & \kappa(x, s_1)\\ \kappa(s_1, s_2) & \kappa(s_1, s_1)\end{vmatrix}\right]ds_1ds_2.$$

On adding

$$\frac{\lambda^2}{2}\rho(x)\int_0^1\int_0^1\begin{vmatrix}\kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix}ds_1ds_2$$

to each side of this equation, the integrand on the right will take the form of a determinant and we get

$$\rho(x)\left[1+\lambda\int_0^1\kappa(s_1, s_1)ds_1+\frac{\lambda^2}{2}\int_0^1\int_0^1\begin{vmatrix}\kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix}ds_1ds_2\right]$$

(10)

$$=\frac{\lambda^2}{2}\int_0^1\int_0^1\begin{vmatrix}\rho(x) & \kappa(x, s_1) & \kappa(x, s_2)\\ \rho(s_1) & \kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \rho(s_2) & \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix}ds_1ds_2.$$

Equations (6) and (10) suggest at once the general formula

$$\rho(x)\left[1+\sum_1^n\frac{\lambda^p}{p!}\int_0^1\int_0^1\cdots\int_0^1\kappa\begin{pmatrix}s_1, s_2, \cdots, s_p\\ s_1, s_2, \cdots, s_p\end{pmatrix}ds_1ds_2\cdots ds_p\right]$$

(11)

$$=\frac{\lambda^n}{n!}\int_0^1\int_0^1\cdots\int_0^1\begin{vmatrix}\rho(x) & \kappa(x, s_1) & \cdots & \kappa(x, s_n)\\ \rho(s_1) & \kappa(s_1, s_1) & \cdots & \kappa(s_1, s_n)\\ \vdots & \vdots & & \vdots\\ \rho(s_n) & \kappa(s_n, s_1) & \cdots & \kappa(s_n, s_n)\end{vmatrix}ds_1ds_2\cdots ds_n,$$

in which

$$\kappa\begin{pmatrix}s_1, s_2, \cdots, s_p\\ s_1, s_2, \cdots, s_p\end{pmatrix}$$

is used to denote the determinant of the pth order in which the element in the ith row and jth column is $\kappa(s_i, s_j)$.

It is not hard to establish equation (11) by induction. More-

over, by using Hadamard's theorem * that the modulus of a determinant of the nth order is less than $M^n\sqrt{n^n}$, M being a number greater than the modulus of every element, it is possible to show that as n increases without limit the right side of equation (11) approaches zero as a limit. Accordingly it follows that in order that equation (4) may have a solution not identically zero it is necessary that

$$D(\lambda) = 0,$$

where

$$(12)\quad D(\lambda)=1+\sum_1^\infty \frac{\lambda^p}{p!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix} s_1, s_2, \cdots, s_p \\ s_1, s_2, \cdots, s_p\end{pmatrix} ds_1 ds_2 \cdots ds_p.$$

As is well known, the convergence of (12) for every value of λ can be established by means of Hadamard's theorem.

Let us now return to equation (3). Guided by the analogy which exists between this equation and a set of non-homogeneous linear equations we may assume that $\rho(x)$ is equal to a fraction of which $D(\lambda)$ is the denominator. Accordingly, if we set

$$\rho(x) = \frac{\sum_0^\infty \lambda^n A_n(x)}{D(\lambda)},$$

and if we substitute in equation (3) we shall find without much difficulty the values of the functions $A_n(x)$; we shall thus obtain Fredholm's formulas.

We can reach the same result more simply, however, by treating equation (3) as we have treated equation (4). From equation (3) we get

$$(13)\quad \rho(x) = -\lambda\int_0^1 \kappa(x, s_1)\rho(s_1)ds_1 + \lambda\int_0^1 \kappa(x, s)\psi(s)ds.$$

As before, we add

$$\lambda\rho(x)\int_0^1 \kappa(s_1, s_1)ds_1$$

to each side of this equation and we obtain an equation analogous to equation (7)

* "Résolution d'une question relative aux déterminants." *Bulletin des Sciences Mathématiques*, 2d series, vol. 17, p. 240 (1893). Cf. W. Wirtinger, "Sur le théorème de M. Hadamard relatif aux déterminants," ibid., 2d series, vol. 31, p. 175 (1907).

$$\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_2) \\ \rho(s_2) & \kappa(s_2, s_2) \end{vmatrix} ds_2 + \lambda\int_0^1 \kappa(x, s)\psi(s)ds. \tag{14}$$

In the second member of this equation let us replace $\rho(x)$, $\rho(s_2)$ by the values given by equation (13). We thus get an equation analogous to equation (8)

$$\begin{aligned}\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] &= -\lambda^2\int_0^1\int_0^1 \rho(s_1)\begin{vmatrix} \kappa(x, s_1) & \kappa(x, s_2) \\ \kappa(s_2, s_1) & \kappa(s_2, s_2) \end{vmatrix} ds_1 ds_2 \\ &\quad + \lambda\int_0^1 \kappa(x, s)\psi(s)ds \\ &\quad + \lambda^2\int_0^1\int_0^1 \begin{vmatrix} \kappa(x, s) & \kappa(x, s_1) \\ \kappa(s_1, s) & \kappa(s_1, s_1) \end{vmatrix} \psi(s)ds_1 ds.\end{aligned} \tag{15}$$

If we treat this equation as we treated equation (8), it is clear that we shall obtain an equation like equation (10) but differing from it by the presence of the two additional terms

$$\lambda\int_0^1 \kappa(x, s)\psi(s)ds + \lambda^2\int_0^1\int_0^1 \begin{vmatrix} \kappa(x, s) & \kappa(x, s_1) \\ \kappa(s_1, s) & \kappa(s_1, s_1) \end{vmatrix} \psi(s)ds_1 ds$$

on the right. Continuing in this way it is not hard to see that we shall get instead of equation (11) an equation differing from it by the presence on the right of the expression

$$\int_0^1 \psi(s)F_n(x, s, \lambda)ds \tag{16}$$

where

$$F_n(x, y, \lambda) = \lambda\kappa(x, y) + \sum_1^{n-1} \frac{\lambda^{p+1}}{p!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix} x, s_1, \cdots, s_p \\ y, s_1, \cdots, s_p \end{pmatrix} ds_1 ds_2 \cdots ds_p. \tag{17}$$

If, in this new equation, analogous to (11), we allow n to increase without limit we obtain

$$\rho(x)D(\lambda) = \int_0^1 \psi(s)F(x, s, \lambda)ds, \tag{18}$$

where $F(x, y, \lambda)$ denotes the limit to which $F_n(x, y, \lambda)$ tends,

$$F(x, y, \lambda) = \lambda\kappa(x, y)$$

$$+ \sum_1^\infty \frac{\lambda^{p+1}}{p!} \int_0^1 \int_0^1 \cdots \int_0^1 \kappa\begin{pmatrix} x, s_1, \cdots, s_p \\ y, s_1, \cdots, s_p \end{pmatrix} ds_1 ds_2 \cdots ds_p. \tag{19}$$

The convergence of this expression can be established by Hadamard's theorem.

If, in equation (18), we replace $\rho(x)$ by its value taken from equation (2), we get Fredholm's solution of equation (1)

$$\phi(x) = \psi(x) - \frac{1}{D(\lambda)} \int_0^1 F(x, s, \lambda)\psi(s)\, ds. \tag{20}$$

It is to be observed that the above demonstration establishes the uniqueness of the solution for every value of λ for which $D(\lambda) \neq 0$.

NEW YORK,
April 1, 1909.

THE CHICAGO SYMPOSIUM ON MATHEMATICS FOR ENGINEERING STUDENTS.

Symposium on Mathematics for Engineering Students. Being the Proceedings of the Joint Sessions of the Chicago Section of the American Mathematical Society and Section A, Mathematics, and Section D, Mechanical Science and Engineering, of the American Association for the Advancement of Science, held at the University of Chicago December 30 and 31, 1907.

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ENGINEERING education, having passed through an infancy somewhat of the "ugly duckling" type in the academic family, and then an adolescence of growth too rapid for either garments or comfort, seems now to have arrived at a stage of maturity in which it recognizes the necessity for heart-searching as to its own real character and habits of life. The demands upon its energies from all quarters are heavy and insistent. It must maintain its vigor and efficiency by ridding itself of every

superfluous ounce of adipose, and by keeping brain, nerves, and muscles in the best condition. Thus the traditional fundamentals of diet and hygiene must be critically scrutinized, and first of all, mathematics.

Mathematics from its antiquity and its necessity has so long been cast into pedagogic moulds that most people cannot separate their notions of it from these moulds — arithmetic, geometry, trigonometry, etc. It has been taught in some form to nearly all civilized persons, and unfortunately, but necessarily, to a great extent by teachers of very slender mathematical endowment — for this endowment is exceedingly rare. Its form and subject matter have been determined by all sorts of considerations of a more or less arbitrary character.

In our own country most of those who took college work in mathematics more than, say, twenty years ago — and this naturally includes most of the participants in the present symposium — were diligently drilled in Todhunter or some dilution of his texts, with little chance of appreciating the real vitality of mathematics or its relations to other living subjects. Those who escaped permanent aversion for it, had by their industry acquired mathematical technique and some power of concentration. This sort of education, while sometimes redeemed by the teacher's personality and doubtless not inferior to much contemporary work in other lines, could not long withstand the influence, on the one hand, of the new generation of mathematicians "made in Germany," and on the other, of competition with the modern scientific or technological subjects. It is safe to say that it hardly now survives to any important extent — except in the minds of certain critics who still describe the conditions of their own youth with fine but unscientific indifference to the progress made in the intervening years. On the other hand, it is but just to grant that the critical revision of a great body of knowledge, and the training of an army of teachers with new points of view — both indispensable steps — have not been matters for a day or a decade. The mere substitution of engineers to teach mathematics would have been vastly simpler, but of course utterly inadequate, not to say futile, as a remedy.

In the accomplishment of this piece of educational evolution, cooperation has been, and will continue to be, essential, and it is as a welcome form of such cooperation that the Chicago symposium, which is here reported, has its chief significance.

The report opens with a valuable description of the "present condition of mathematical instruction for engineers in American colleges," based on a study of seventeen selected institutions, by Professor Townsend, of Illinois. There is a judicious deprecation of undue advance of entrance requirements in mathematics, and of undue emphasis on uniformity. The statistical tables of entrance requirements and of time allotment in course are naturally to be used only with caution for purposes of comparison, and with due regard for Poincaré's dictum, that mathematics — a fortiori statistics — is the art of calling different things by the same name. Interesting references are made to the desirability of elective mathematics, to the positive and negative value of the "Perry movement," to recent tendencies to deal with mathematics as one subject rather than as many, etc. It is wisely urged that what is best for the engineering student in the first two years of his mathematical work is also best for the student who is taking mathematics as an element in a liberal education. In view of the emphasis put by some other speakers on the use of mathematics as a "tool," it may be pertinent to supplement the remark just quoted by adding that the best engineering schools and departments aim to combine in a truly liberal sense general and professional education, and that any attempt to teach mathematics merely as a tool would, to the extent of its dubious success, mean sacrificing the first of these objects. One of the important elements in higher — as distinguished from merely technical — education is surely the student's relation with teachers of widely different points of view, mathematical, philosophical, literary, etc., as well as scientific and engineering. It is perhaps natural that the same critics who see in mathematics nothing but a machine into which data are fed, and from which results are ground out, should most insist that the subject ought to be taught as a tool.

The symposium includes further a paper on conditions in foreign countries (chiefly Germany) by Professor Ziwet of Michigan, general papers by Mr. C. F. Scott of the Westinghouse Company and Dr. R. S. Woodward of the Carnegie Institution, a presentation of the standpoint of the practicing engineer by Mr. Ralph Modjeski of Chicago, and Mr. J. A. L. Waddell of Kansas City; of the standpoint of the professor of engineering by Professors Williams of Michigan, Talbot of Illinois, and Swain of the Massachusetts Institute of Technology; of

the standpoint of the professor of mathematics, by Professors Slichter of Wisconsin, and Woods of the Massachusetts Institute of Technology, and by President McNair of the Michigan College of Mines.

For the readers of this BULLETIN it may naturally be most profitable to consider the criticisms of the engineers—including Mr. Scott—and the professors of engineering. Of the three engineers, one who received his mathematical training abroad and naturally expresses himself with some reserve in regard to American conditions, looks forward to the time when only the applied mathematics will be taught in college, all necessary abstract mathematics forming a part of the entrance requirements. The other two dwell on unsatisfactory elements in existing conditions, and on results to be sought, both of them emphasizing the importance of skill in the use of mathematics as a tool. This view, urged still more emphatically by one of the professors of engineering, might remind an unsympathetic reader of the ingenuous evidence so often presented to the Committee on Ways and Means by beneficiaries of the protective tariff. As a matter of course, the mere use of mathematics as a tool can not be most successfully taught by mathematical specialists. Exactly this is, and must always be, one of the most important functions and opportunities of the teachers of physics, mechanics, and engineering, as was naturally insisted in the course of the discussion by the mathematical speakers. The mathematicians must build a foundation adapted to the educational superstructure. With this their responsibility ends. If they permit themselves to be diverted from this responsibility for laying a solid foundation, it needs no engineer to foretell the disastrous consequences.

The further criticism of the engineers and the professors of engineering, while varying widely in details, seems to connect itself with the general notion that mathematics is habitually presented to engineering students in too abstract a form, with too little attention to the statement of problems and the interpretation of results, and with too little appreciation of the uses to be made of mathematics in engineering work. There is still unquestionably a measure of truth in this criticism, but for reasons which have been already indicated, the extent to which the criticism is valid can be determined for each institution only by a careful investigation of what it is now doing. No one of the speakers in question mentions having made such an

investigation, even in a single institution. One of the engineers remarks that "possibly methods have changed of late years, but nothing that the writer has seen or heard indicates to him that any fundamental improvement has been effected." One of the professors of engineering states that in his classes he has had students from most of the principal colleges and technical schools of the country, and that he has failed to notice any great difference in them in respect to lack of mathematical power. Such statements hardly suggest that degree of careful discrimination which may be fairly expected of a scientific critic on a scientific subject.

The criticism that teachers of mathematics neglect the applications, so that their students do not learn to *do* anything, suggests the inquiry: Just what do they need to learn to do? In the large engineering institutions and departments the mathematical instruction must often be given to classes made up of students from several courses, or departments, for whom the number and variety of applications of common interest will naturally be limited. Moreover, the stronger engineering schools draw a considerable — and probably an increasing — proportion of their students from academic institutions, in which mathematics has been taught necessarily without specialization. Under these conditions all that can fairly be demanded of the teachers of mathematics is that they articulate their work as closely as may be with the dependent physics and mechanics which immediately succeed it, by including physical and mechanical problems of general interest.

The plea of several speakers that geometry be more emphasized, and that descriptive geometry made more mathematical, deserves careful attention. This question is, of course, to a considerable extent one of organization rather than of mathematical teaching per se. Any thorough-going improvement would seem to imply that descriptive geometry, and perhaps mechanical drawing, should be given under the direction of the mathematical department as its laboratory work, rather than independently, as is now apt to be the case. Undoubtedly teachers of mathematics would welcome any tendency on the part of their colleagues having charge of these outlying subjects to give them a more mathematical character.

It is not without interest that the most severe criticism of the present teaching and teachers of mathematics is coupled with a citation of Sir William Hamilton as an authority on the

value and the pedagogy of mathematics. In view of the vogue of Hamilton's criticism, and the limited publicity of Professor Keyser's recent exposé of it, a brief quotation may be made from his Columbia University lecture on Mathematics. In Hamilton's article "the reader is apparently confronted with the assembled opinions of the learned world, and — what is more amazing — they all agree. Literati of every kind, of all nations and every tongue, orators, philosophers, educators, scientific men, ancient and modern, known and unknown, all are made to support Hamilton's claim, and even the most celebrated mathematicians seem eager to declare that the study of mathematics is unworthy of genius and injures the mind. . . . The Scotchman's victory was complete, his fame enhanced, and his alleged judgment regarding a great human interest of which he was ignorant has reigned over the minds of thousands of men who have been either willing or constrained to depend on borrowed estimates. But even all this may be condoned. Jealousy, vanity, parade of learning, may be pardoned even in a philosopher. Hamilton's deadly sin was none of these, it was sinning against the light." It has been shown by Bledsoe and Pringsheim "that Hamilton by studied selections and omissions deliberately and maliciously misrepresented the great authors from whom he quoted — d'Alembert, Blaise Pascal, Descartes and others — distorting their express and unmistakable meaning even to the extent of complete inversion." Even a philosopher thus discredited may have told some truths of his own, but his opinion can hardly be deemed to add much weight to the present criticism. Any attempt to discuss the general educational value of mathematics would, however, far exceed the limits of this review, besides being presumably unnecessary for its readers.

It should not be inferred from any thing that has been said that adverse criticism of mathematics, or mathematicians, was general on the part of the engineering speakers. It is too often the extreme criticism — picturesque rather than accurate — which impresses even a professional public, but the symposium as a whole was rather notable for agreement in fundamentals. As to the importance of the teacher's training and continued interest in applied mathematics, as to the need of emphasis on concrete problems, as to the relatively low value of lectures and mathematical blackboard work, there appears to have been no difference of opinion. It may be added, in connection with the

training of teachers of mathematics, that there would be many advantages if a larger share in that training could be taken by the technological schools and departments than is at present the case.

The case for the teachers of mathematics was ably sustained by Professors Slichter and Woods. The former from the standpoint of a consulting engineer as well as that of a teacher of mathematics, emphasizes the increasing need on the part of technology for more and better science, and insists that mathematics is a basal science rather than a tool. Professor Woods dwells particularly on the urgent need that teachers in various subjects dependent on mathematics continue and complete the work which the mathematicians can only have begun. Mention may also be made of the valuable contribution to the subsequent discussion by President Howe of the Case School.

The pamphlet closes with the announcement of a committee of fifteen — the actual list seems to include twenty names — to make to the Society for the Promotion of Engineering Education such a report on mathematics for colleges of engineering as in their opinion will be of service to teachers in such institutions.

After all, however, the real value and importance of such a symposium consists in the intangible influence of the discussion itself on its hearers and readers. Reports and resolutions are at the best merely the stamp on the coin, and too often make but a faint impression thereon. The teaching of mathematics to students of engineering is still in that stage of development by individual initiative in which diffusion of information and circulation of ideas are invaluable, but in which ideas formulated in one year as novel may in the next seem commonplace. Up to a certain point it is easy to agree as to principles; beyond that point efforts at agreement may be for the time quite futile. What is difficult, yet possible and necessary, is a constant improvement of our individual practice. It would be a great service — but an exceedingly difficult one — for any committee or any individual to bring the criticisms of current mathematical teaching to the test of a searching statistical investigation, the only conclusive test. It is also a great but a quite practicable service to prepare for the use of teachers and students of mathematics texts involving more numerous and more varied applications. In this direction the teachers of mathematics have done much and need all support and cooperation — criticism also, if it be con-

structive, or, at least, discriminating. The need of improvement in line with the constructive criticism is, in the judgment of the present reviewer, freely recognized and continually better met.

H. W. TYLER.

OSGOOD'S CALCULUS.

A First Course in the Differential and Integral Calculus. By WILLIAM F. OSGOOD, PH.D., Professor of Mathematics in Harvard University. New York, the Macmillan Company, 1907, pp. xv + 423. Revised edition, 1909, pp. xv + 462.

PROFESSOR OSGOOD in his presidential address before the AMERICAN MATHEMATICAL SOCIETY* has discussed and illustrated the principles which his experience has led him to consider should govern the teaching of the calculus. In the present text he gives us the detailed application of those principles to the difficult pedagogical problems which confront the instructor in the first course in this subject.

Successful instruction in mathematics requires careful adjustment of the conflicting claims of rigor, formalism, and interest. Professor Osgood has recognized† that rigor is a relative matter particularly in elementary instruction, and has enunciated the principle that in such instruction a discussion is to be regarded as rigorous if it meets all the logical demands which the student can be regarded as capable of appreciating at that time. This principle is at bottom the same as that which governs contemporary judgment of productive work, and its application to instruction is but a recognition of the fact that the mathematical development of the individual differs in general from that of the race at most by a transformation of similarity.‡ It is evident, however, that such a principle must be applied with care, for otherwise it may be cited in defense of a multitude of mathematical sins. If used with judgment, however, as is the case in this text, it becomes the very foundation of successful mathematical teaching.

The applications of this principle are in evidence throughout the book. For example: the theorem on the limit of the sum is at first (page 12) tacitly assumed, then (page 14) mentioned in a footnote, and finally proved (page 15) when the progress

* BULLETIN, vol. 13 (1907), pp. 449–467.
† *Annals of Mathematics*, ser. 2, vol. 4 (1903), p. 178.
‡ Cf. Cantor, Geschichte der Mathematik, 3 Aufl., Bd. 1, p. 3.

of the work has rendered evident its necessity. Again in the discussion of $D_x(u+v) = D_x u + D_x v$ and similar theorems (pages 14, 21) the existence of $D_x u$ and $D_x v$ is assumed without comment, but later (page 423) the assumption and its significance are discussed. A similar point is illustrated in the treatment of $D_x y = D_t y / D_t x$ (pages 23, 423).

It is in conformity with this principle too, that Professor Osgood, an authority on questions of uniform convergence, has omitted this important subject from his text. The omission is at first sight somewhat startling in view of the author's tendencies, and yet when one reflects that the genius of a Weierstrass was required to establish uniform convergence as a constructive principle, the wisdom of omitting the subject from a course for beginners is apparent. As a consequence of this omission the continuity, differentiation, and integration of series are merely mentioned with a reference to the author's Infinite Series.

Progressive criticism of results already obtained and of methods already used must form an important part of a system of instruction which continually modifies its standard of rigor to accord with the student's increasing power of comprehension. This criticism is the more valuable to the student in proportion as it is constructive rather than destructive. It is therefore desirable that the necessarily incomplete discussions of an elementary course should be susceptible of ready modification in order to satisfy later more exacting demands, but they should not be such as to require extensive reconstruction. A case in point here is the treatment of Duhamel's theorem (page 164). The theorem is given in the original inexact form in which the question of uniformity is so veiled that in order to bring out the precise point at issue considerable destructive criticism of the type which Professor Osgood himself has given * is rendered necessary. Professor Osgood's own revised formulation † is as simple in statement and proof as the original, if the question of uniformity is waived, and it has the important advantage of needing only to be supplemented rather than reconstructed in a later and more advanced course. The revised form has, moreover, the further advantage of more direct application to the problem wherein the theorem is most valuable, namely, that of the formulation of questions involving the definite integral as the limit of a sum.

* Osgood, *Annals of Mathematics*, ser. 2, vol. 5 (1903), p. 169ff.
† Ibid., p. 173

The formal and manipulative side of the calculus has till recently received principal emphasis in American texts, and indeed to such an extent that the student has too frequently failed to obtain an adequate conception of the fundamental ideas of the subject and their application to concrete problems.

On these fundamental ideas and methods rather than on mere manipulation the stress is laid in Professor Osgood's text. For example an extended graphical and analytical treatment of $D_x x^n = nx^{n-1}$ (n fractional and negative) is given; the existence of $e = \lim_{n=\infty} (1 + 1/n)^n$ is accurately though simply established, and the idea of the definite integral as the limit of a sum is developed with careful attention to detail by means of a series of geometrical and statical problems. On the other hand in place of the usual treatment of the integration of rational fractions, of $\int R(x, \sqrt{ax^2 + bx + c})dx$, and of $\int x^m(a + bx^n)^p dx$, there is a short introduction to the use of the well known integral tables of B. O. Peirce. The most serious objection to this departure from the conventional treatment lies in the fact that indefinite integration, as an inverse operation, is always a difficult process for the student, and the power to reduce a given function to a known integrable form, *e. g.*, to one of those in the tables, is in general gained only by considerable directed practice. The traditional discussion of the types above mentioned may well be displaced by an introduction to the use of the tables, but in the opinion of the reviewer that introduction should contain a somewhat detailed discussion of the simpler integrable forms; a statement concerning forms not in general integrable, *e. g.*, $\int R(x, \sqrt{ax^3 + 3bx^2 + 3cx + d})dx$; and a statement at least of the theorems concerning the integrability of $R(x)$, $R(x, \sqrt{ax^2 + bx + c})$, $R(\sin x, \cos x)$, etc.

Abundant opportunity for practice and differentiation and integration is afforded by the supplementary exercises (revised edition, pages 324–359).

The space and time saved in the chapter on integration has been used to most excellent advantage in that on partial differentiation.

In much of the work of the elementary course in the calculus the analytical difficulties are those arising in the proofs of the theorems used. Here the trouble is almost invariably one of interpretation. The meaning of the partial derivative especially when several dependent and independent variables occur is sel-

dom understood. The important section (page 306) on the notation used for partial derivatives, and the problems 4, 8, 21, 24, pages 312–315 form a much needed addition to the usual treatment of this subject, and will be a welcome aid to the teacher and student of thermodynamics. Were such discussions more common in texts on the calculus, their proper place, they would be unnecessary in treatises on thermodynamics.* The treatment of the various cases of $du=\partial u/\partial x\,dx+\partial u/\partial y\,dy$, etc. (page 292ff.), is another valuable aid to the clarification of the student's ideas concerning partial differentiation, and together with the treatment of the differential of a function of a single variable (page 92ff.) applies the principles concerning differentials enunciated in the author's presidential address† in such a way as to leave no opportunity for the "little zeros" to insinuate themselves into the student's mind.

Professor Osgood has long emphasized by precept and example the importance of developing new mathematical concepts in the student's mind by means of problems, *i. e.*, of causing the new mathematical idea to appear as a necessary element for the solution of a definite geometrical or physical problem.

His chapter on the definite integral as the limit of the sum is an example of this method, as well as of gradual development of precision in concept and demonstration conformably to the student's advance in assimilative power. The important purely theoretical matters in such a chapter are the concept of the definite integral as the limit of a sum, the upper and lower integrals, the fundamental theorem of the integral calculus, the proof of the existence of the definite integral, and the theorem of Duhamel. In Professor Osgood's treatment the area problem leads (page 153) to the concept of the definite integral as the analytic formulation of a limit the existence of which is geometrically evident. A previous discussion (pages 111–2) had developed the relation $d\,(\text{area})/dx = \text{ordinate}$. Comparison gives the fundamental theorem of the integral calculus. Of course the existence of an area is tacitly assumed, but that is at this stage of minor importance. The essential thing is that the student visualize the definite integral and understand its relation to the anti-derivative, *i. e.*, that he recognize the first as formulating his problem, the second as a means of evaluating its result, and shall see the necessity of the fundamental theorem

* E. g., Bryan, Thermodynamics, Chapter III, pp. 21–26.
† BULLETIN, loc. cit., p. 451.

of the integral calculus as the link which connects his method of formulation with his method of evaluation.

The next step is the introduction of the upper and lower integrals.* The problem of the fluid pressure upon a vertical plane surface with curvilinear boundary leads to the establishment of the element of the upper and lower integrals, their relation to the element of the integral and the passage to the limit which shows the existence of the upper and lower integrals, their equality and the existence of the integral. Here the reasoning depends on geometrical and statical considerations.

The way is now prepared for the analytic proof of the existence theorem for the definite integral. The proof is given in the usual way for a monotonic function. Though it is illustrated by the figure of the circumscribed and inscribed areas, and the language is in part geometrical, no fact is used the arithmetic truth of which is not immediately evident from the definition of the monotonic function. The extension to the case of a finite number of maxima and minima is then made, and finally in the appendix (page 423) attention is called to the case of functions with an infinite number of maxima and minima.

The chapter is noteworthy for the manner in which the difficulties of the theory of the definite integral are brought to the student's attention as inherent in the problems to which the integral calculus is applicable.

The chapters on double and triple integrals follow the same lines and there is the same gradual development of the fundamental concepts and avoidance of unnecessary analytical difficulties. The insistence on the distinction between the double (or triple) integral and the iterated integral, and on the fundamental theorem of the integral calculus as the link which joins them, is as desirable as it is unusual in elementary texts.

Few students, even those for whom mathematics is an elective rather than a required subject find their chief interest in the precision of its concepts, the rigor of its demonstrations or the elegance of its formulas. They continually demand immediate application to concrete problems as a reason for considering the subject at all.

Professor Osgood's treatment is well fitted to satisfy this demand. For example, as soon as the derivative has been defined

*The terms upper and lower integrals are not mentioned, but the concept is used.

and a few rules of differentiation established, the new method is applied to problems of maxima and minima, of velocity, of curve tracing, and of roots of equations in such a way as to make evident its power as compared with that of elementary methods. The result is that the first three chapters would by themselves form an excellent introduction to the differential calculus for those students who desire a knowledge of its scope and methods without going into analytical details.

The earlier American texts in the calculus have in general confined their applications to the field of geometry. The adherents of the "Perry movement" have gone to the other extreme and have produced a flood of problems taken from engineering practice. They have, however, too often forgotten that the student for whom the problems are designed has not yet acquired the technical knowledge necessary for an appreciation of their meaning and importance.

Professor Osgood has by no means neglected geometry, as his chapters on the cycloid, on curvature and evolutes, on envelopes, on partial differentiation, and on definite integrals bear witness. On the other hand, he has throughout the text, and especially in the chapter on mechanics, provided numerous problems illustrating the application of the calculus to physical phenomena, and that without requiring more technical knowledge than can be introduced into the text without distracting the student's attention from the true aim of the work — the application of the calculus to physical phenomena, as opposed to the formulation of a particular physical problem.

In conclusion it may be said that Professor Osgood's text is characterized by insistence on the real essentials of the calculus, and by consistent maintenance of that close relation between theory and application to which both pure and applied mathematics owe their most important advances.

CHARLES N. HASKINS.

THE UNIVERSITY OF ILLINOIS,
March, 1909.

SHORTER NOTICES.

Grundlehren der neueren Zahlentheorie. By PAUL BACHMANN. Leipzig, G. J. Göschen, 1907. Sammlung Schubert, v. 53. 11 + 270 pp.

KLEIN has recently * called attention to the two fundamental types of mathematical development. The one is concerned primarily with the exposition of a given branch of mathematics for its own sake and with its own methods. If this type alone obtained, mathematics would appear as a series of more or less distinct theories — algebra, trigonometry, calculus, etc. — which may show here and there incidental points of contact, but which are not organically connected. The other type of development, on the other hand, is concerned with just this welding together of the various so-called branches into a unified whole. It conceives, for example, the two great divisions of analysis and geometry as being only two different aspects of the same thing. Wherever this second type of development has made itself prominently felt it has meant a gain in power; not simply by giving various interpretations to the same theorem, but, and perhaps chiefly, by making the methods of one theory available for research in another.

All branches of mathematics have felt the influence of this second type of development to a greater or less degree. Even the theory of numbers, which for a while held a certain autonomy in its methods, is at present in its more advanced portions at least in intimate union with other fields, *e. g.*, the theory of functions of a complex variable. In recent years, however, even the elementary portions of the theory of numbers have had to submit to the introduction of matter and methods from another field; viz., that of geometry. To speak of a "modern" treatment of the elementary theory of numbers brings to mind at once the lectures of Klein — that master exponent of the second type of development — during the year 1895–96, and Minkowski's work on the Geometrie der Zahlen. An elementary text on the theory of numbers which makes consistent use of these geometric interpretations and methods where they are available is obviously a great desideratum. It was therefore with much joyful anticipation that the reviewer opened

* Elementarmathematik vom höheren Standpunkte aus, pp. 180–187.

this recent addition to the Sammlung Schubert. The preface seemed to intimate that the author intended to write just such a text as we had hoped for. But after reading the little book with much occasional pleasure the reviewer laid it down with a distinct feeling of disappointment. Of geometric interpretation there is a little; of geometric method there is none.

Having thus expressed our disappointment regarding one feature of the text, it is a pleasure to mention some that are pleasing. The notion of a domain of rationality — or field — is introduced on the first page, and that of a modulus — domain of integrality — on the second. The latter is defined as any set of numbers which is closed under the operations of addition and subtraction; *i. e.*, such that if a and b are numbers of the set, $a + b$ and $a - b$ are also in the set. Thus, if $a, b, c, \ldots$ are any given numbers, the set of all numbers of the form $ax + by + cz + \ldots$, where x, y, z, are any integers, forms a modulus which is denoted by $[a, b, c, \ldots]$. This notion is made fundamental throughout the book; a fact which differentiates it from the ordinary treatment, and which introduces a refreshing unity of conception. Many results are stated in terms of it which are usually stated differently. For example, two integers n, n' are said to be congruent mod m if $n - n'$ is a number of $[m]$. Its advantageous use in the development of Dedekind's theory of ideals is obvious.

The book is divided into two parts devoted respectively to "the rational number field" (157 pages) and to "the quadratic number field" (113 pages). The first four chapters discuss briefly and sometimes with a refreshing novelty the usual topics relating to 1) the divisibility of numbers, 2) congruences, 3) quadratic residues, including the law of reciprocity, 4) the linear form $f = ax + by$. Another unifying principle is found in the fact that the theorem concerning the solvability in integers of the equation $ax + by = 1$ is made fundamental throughout this portion of the text. Chapter 4 contains a complete discussion of the equivalence of moduli $[a, b]$, which is of fundamental importance later in the discussion of ideals. Here also we have for the first time a geometric interpretation of the results by means of a lattice (Zahlengitter). We are sorry that in this chapter the author did not see fit to include also Klein's elegant geometric interpretation of the development of an irrational number into a continued fraction. Chapter 5, which is the last of Part 1, begins the discussion of the equivalence of

quadratic forms, the discussion not being completed until later in connection with the theory of quadratic numbers.

The author has developed the theories of quadratic forms and quadratic numbers in conjunction in Part 2, each supplementing the other and forming together a single theory. In this fact he finds his chief reason for calling his treatment "modern." The result is indeed "esthetically satisfying" (quotation from the preface). In view of the fact, however, that the book is professedly intended for beginners, it may perhaps be doubted whether this treatment is pedagogically desirable. There can be no manner of doubt that the beginner will find Part 2 hard reading; and it does not appear evident that the intermingling of forms and ideals, however beautiful the result, makes the reading any less difficult.

The book is remarkably free from typographical errors. Besides the single one noted in the corrigenda, the reviewer has noticed only two; one on page 36, 3d line from below, where *nr.* should read Nr.; and one on page 70, line 7, where the reference should be to Nr. 4 instead of to Nr. 3. In this connection we may note further that on page 25 the expression $\phi(1)$ must be defined as equal to unity, and on page 71 the condition $\Delta \neq 0$ should be added.

J. W. YOUNG.

The Axioms of Descriptive Geometry. By A. N. WHITEHEAD. Cambridge University Press, 1907. viii + 74 pp.

THIS little volume is No. 5 of the Cambridge Tracts in Mathematics and Mathematical Physics. It follows a previous tract (No. 4 of the same series) by the same author, on the Axioms of Projective Geometry, to which constant reference is made. The work begins with formulations of the axioms, those of Peano and Veblen being given in detail. Chapter II treats of the relation of projective space and the associated space obtained by taking a convex region of the projective space, such a convex region being shown to be a descriptive space. The development follows that of Bonola closely. Chapter III contains the development of ideal elements in a descriptive geometry, the work being drawn from Veblen, apparently. In Chapter IV a "General theory of correspondence" is introduced through the medium of projective coordinates, the ideas of continuous groups of projective transformations and their infinitesimal transformations being developed from analytic

geometry. Chapters V–VII contain a logical analysis of the method of superposition as applied to geometrical proofs. After the axioms of congruence of Pasch, the development of congruence in terms of previous concepts is given, after Lie, by consideration of congruence groups and their subgroups. Metrical geometries are then treated with reference to the absolute, and the kind of a metrical geometry obtained from the congruence group is shown to depend upon an assumption concerning the nature of the absolute.

The proofs are usually only sketched and the reader is left to complete the details. The work could be improved by a more formal statement of the theorems and the exact group of axioms upon which each theorem depends. For the reader who has some knowledge of the general methods used and the point of view, the book should serve as a very useful compilation of results from many sources.

F. W. OWENS.

Lectures de Mécanique. Première Partie. By E. JOUGUET. Paris, Gauthier-Villars, 1908. x + 210 pp.

THIS book is based upon the idea that "it is very instructive for one who wishes to comprehend the nature of the principles and laws of mechanics to follow the history of their development." The subject matter is presented mainly by long quotations from the original authors, interspersed and followed by comments of the writer, who believes it is interesting to study the classic writers on mechanics as one studies the classic writers in literature. The quotations occupy about 163 of the 210 pages in the text. M. Jouguet has in view the physical aspect of the subject and limits himself to the fundamental principles and essential laws. The first part is devoted to the beginnings of mechanics. After a brief introduction on the mechanics of the ancients there are three chapters on statics and four on dynamics.

The chapters on statics are devoted to the lever, the parallelogram of forces, and the principle of virtual work. The conception of each physical law is traced in the problems which have suggested it, the final statement of the law being given in the words of the author who first formulated it definitely. For example the first suggestion of the principle of virtual work is found in Aristotle's treatment of the lever. The evolution of ideas leading to the statement of the principle is then

traced by quotations from the writings of Leonardo da Vinci, Cardan, Galileo, and others, its final formulation being given by Descartes.

The first chapter on dynamics contains long quotations from Galileo and Descartes presenting the idea of inertia. The second chapter deals with the phenomena of direct impact from which are deduced the law of action and reaction and the determination of the velocity after collision. The last two chapters treat of the center of oscillation, force and acceleration, and kinetic energy.

Throughout the book attention is called to the different ways of reaching a conclusion, the contrast being particularly striking in the development of the law of inertia, where the method of Descartes is characterized as metaphysical while the inferences of Galileo are drawn from physical observations and experiments. Some of the false conclusions of the early thinkers are presented in connection with the correct results which have led to the statement of modern fundamental principles.

W. R. LONGLEY.

Cours d'Astronomie. Seconde Partie: *Astronomie pratique.* By H. ANDOYER. Paris, A. Hermann & Fils. 1909. 299 pp.

THIS volume completes the course, of which the first part deals with theoretical astronomy. As a text-book for a first course in practical astronomy, the subject matter of the second part is well chosen. Only the common instruments, namely, the theodolite, the equatorial, and the meridian transit are treated in detail, the more special instruments, such as the heliometer and the siderostat, receiving very brief mention. No cuts of astronomical instruments appear in the book and the question of construction is dismissed with the remark that a few hours' acquaintance with the instrument itself is more profitable than a study of the most minute description. The theory of the constants and errors of the instruments and their adjustment is thoroughly worked out. Very naturally the problems which receive the most attention are those connected with the determination of the geographical position of the observer, although a rather brief discussion is given of the observations used in the determination of the fundamental constants, including refraction, aberration, nutation, precession, obliquity of the ecliptic, position of the vernal equinox, and parallax of moon, sun, and stars. In addition to the subject

matter of observational astronomy the book contains an introductory chapter on numerical calculation, including the theory of interpolation and the method of least squares. The last chapter is an exposition of Gauss's method for determining the elements of an elliptic or parabolic orbit from three complete observations.

The material in the book is not new, but its careful selection and clear presentation make it valuable to the instructor of beginning classes in practical astronomy.

W. R. LONGLEY.

NOTES.

AT the Colloquium of the AMERICAN MATHEMATICAL SOCIETY, to be held at Princeton University, September 15–17, courses of lectures will be delivered by Professor G. A. BLISS, on "Fundamental existence theorems," and Professor EDWARD KASNER on "Geometric aspects of dynamics." Professor J. H. JEANS having resigned his position at Princeton, the course of lectures announced to be given by him has been cancelled.

THE seventy-ninth annual meeting of the British Association for the Advancement of Science will be held in Winnipeg during the week August 25 to September 1. Sir J. J. THOMSON is president of the association for this meeting and Professor E. RUTHERFORD is president of Section A (Mathematics, Physics and Astronomy). Low rates are quoted by the Canadian railroads and there will be a number of excursions including one to the Pacific coast. Abstracts of papers, which the authors desire to have printed for use at the meetings, should be in the hands of the Secretary of Section A, Professor J. C. FIELDS, University of Toronto, early in June. For information other than that connected with the programme, application should be made to the "Local Secretaries of the British Association," Winnipeg, Manitoba.

AT the meeting of the London mathematical society held on April 22 the following papers were read: By F. TAVANI, "The general principles of the theory of integral equations"; by H. R. HASSÉ, "The equations of electrodynamics and the null influence of the earth's motion on optical and electrical

phenomena"; by S. N. WATSON, "The solution of a certain transcendental equation"; by H. BATEMAN, "The physical applications of certain conformal transformations of a space of four dimensions and the representation of a space time point by means of a sphere"; by A. E. WESTERN, "Some criteria for the residues of eighth and other powers"; by W. H. YOUNG, "On the discontinuities of a function of one or more real variables."

KING LEOPOLD, of Belgium, offers a prize of $5,000 to the author of the best work on the subject: "Describe the progress of aerial navigation and the best means to encourage it." Competing memoirs may be written in any of the better known languages of Europe. They should be sent to the minister of sciences and arts at Brussels before March 1, 1911.

THE council of the American federation of teachers of the mathematical and the natural sciences met in Baltimore December 28, 1908. Reports were heard from the local organizations, and a memorial presented to Congress, urging that the board of education be given more extensive power. Professor H. W. TYLER, of the Massachusetts Institute of Technology, was elected president, and Professor C. R. MANN, of the University of Chicago, secretary-treasurer for the present year.

The following advanced courses in mathematics are announced for the year 1909–1910:

HARVARD UNIVERSITY. — Each course is three hours per week. — By Professor W. E. BYERLY: Introduction to modern geometry and modern algebra; Advanced calculus; Trigonometric series (with Professor Peirce). — By Professor W. F. OSGOOD: Theory of functions, I; Linear differential equations. — By Professor C. L. BOUTON: Theory of numbers (first half year); Elementary differential equations (second half year); Geometric transformations. — By Professor J. K. WHITTEMORE: Differential geometry of curves and surfaces (first half year); Properties of polynomials and invariants (second half year); Celestial mechanics (first half year). — By Professor J. L. COOLIDGE: Probability; Algebraic plane curves. — By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics (second half year). — By Dr. H. N. DAVIS: Elements of mechanics. — By Mr. G. C. EVANS: Vector analysis

and quaternions. Courses of reading and research are offered by Professors Byerly, Osgood, Bouton, Whittemore, and Coolidge. A seminary in geometry will be conducted by Professors Bouton, Whittemore, and Coolidge.

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Higher geometry, two hours; Theory of functions, two hours (first half year); Vector analysis, two hours (second half year). — By Dr. A. COHEN: Differential equations, two hours; Elementary theory of functions, two hours. — By Dr. A. B. COBLE: Theory of groups, two hours; Theory of probabilities, two hours (first half year).

UNIVERSITY OF PENNSYLVANIA (Summer School, July 6 to August 17, 1909). Each course one hour daily. — By Professor I. J. SCHWATT: Infinite series and products. — By Professor G. H. HALLETT: Modern analytic geometry. — By Professor F. H. SAFFORD: Differential equations. — By Dr. O. E. GLENN: Invariant algebra.

PROFESSOR H. E. TIMERDING, of the University of Strassburg, has accepted a professorship of mathematics at the technical school of Braunschweig.

DR. W. VOGT has been appointed docent in mathematics at the technical school at Karlsruhe.

Dr. W. KUTTA, of the technical school at Munich, has accepted an associate professorship of mathematics at the University of Jena.

AT the meeting of the National academy of sciences held at Washington April 22–24 the following mathematicians and physicists were elected to membership: J. S. AMES, of Johns Hopkins University; MAXIME BÔCHER, of Harvard University; OSKAR BOLZA, of the University of Chicago; H. CREW, of Northwestern University.

PROFESSOR W. F. OSGOOD, of Harvard University, has been elected corresponding member of the Mathematical Society of Charkow.

DR. C. L. E. MOORE, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

AT Williams College Dr. F. L. GRIFFIN has been promoted to an assistant professorship of mathematics.

MR. L. L. SILVERMAN, of the University of Missouri, has been appointed instructor in mathematics at Cornell University.

PROFESSOR MAXIME BÔCHER, of Harvard University, has been granted a leave of absence for the academic year 1909–1910; he will spend the year in study in Europe.

PROFESSOR H. v. STAHL, of the University of Tübingen, died April 6, at the age of 65 years.

RECENT second-hand catalogues of mathematical works: A. Hermann & Fils, 6, rue de la Sorbonne, Paris, Bulletin No. 97, about 1,500 titles.—Conrad Skopnik, Prinz Louis Ferdinandstrasse 1, Berlin, N. W., Catalogue No. 43, 731 titles in mathematics, physics, and astronomy.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BÜCHEL. See MITTEILUNGEN.

FINKE (P.). Ueber Scharen von ∞^5 Kurven im gewöhnlichen Raume. (Diss.) Berlin, Frenkel, 1909. 8vo. 36 pp. M. 2.00

GROSS (T.). See STURM (C.).

HOFFMANN (F.). Der Satz von Fermat. Strassburg, 1909. 8vo. M. 0.50

HOPPE. See MITTEILUNGEN.

MELLOR (J. W.). Higher mathematics for students of chemistry and physics. London, Longmans, 1909. 8vo. 664 pp. Cloth. 15s.

MITTEILUNGEN der mathematischen Gesellschaft in Hamburg. Vol. IV. 9tes Heft. Redigiert von Hoppe, Büchel und Umlauf. Leipzig, Teubner, 1909. 8vo. Pp. 403–456. M. 1.60

OSGOOD (W. F.). Differential and integral calculus: a first course. Revised edition. New York, Macmillan, 1909. 12mo. 15 + 462 pp. $2.00

STURM (C.). Lehrbuch der Analyse (Cours d'analyse). Uebersetzt von T. Gross. Neue Ausgabe. Berlin, 1909. 8vo. M. 10.00

STURM (R.). Die Lehre von den geometrischen Verwandtschaften. Dritter Band: Die eindeutigen linearen Verwandtschaften zwischen Gebilden dritter Stufe. Leipzig and Berlin, Teubner, 1909. 8vo. 8 + 574 pp. Cloth. M. 20.00

UMLAUF. See MITTEILUNGEN.

VLACHOS (C.). Der Beweis des Fermat'schen Satzes. Berlin, Gottheiner, 1908. 8vo. 7 pp. M. 0.75

VOGT (H.). Eléments de mathématiques supérieures à l'usage des physiciens, chimistes et ingénieurs et des élèves des faculté de sciences. 5e édition. Paris, Vuibert, 1909. 8vo. 8 + 708 pp.

II. ELEMENTARY MATHEMATICS.

DONADT (A.). See LÜBSEN (H. B.).

DURELL (C. V.). A course of plane geometry for advanced students. Part I. New York, Macmillan, 1909. 8vo. 11 + 219 pp. $1.50

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. Zum Gebrauche für Schule und Praxis bearbeitet. 100ste Auflage. Halle, Strien, 1909. 8vo. 2 + 176 + 35 pp. Cloth. M. 2.50

HUPE (A.). See MÜLLER (H.).

LESSER (O.). See SCHWAB (K.).

LORMEAU (E.) et MERLET (P.). Problèmes de géométrie donnés aux examens du baccalauréat, latin-sciences, sciences-langues, mathématiques A (1904–08), énoncés en la géométrie André et Lormeau. Enoncés et solutions. Paris, André. 8vo. 279 pp. Fr. 4.00

LÜBSEN (H. B.). Ausführliches Lehrbuch der Elementar-Geometrie. Zum Schul- und Selbstunterricht mit Rücksicht auf die Zwecke des praktischen Lebens in 2 Teilen bearbeitet. 30ste Auflage, neubearbeitet von A. Donadt. Leipzig, Brandstetter, 1909. 8vo.
1. Ebene Geometrie. 8 + 176 pp. M. 2.80
2. Körperliche Geometrie. Mit einer Anleitung zum perspektiven Zeichnen. 2 + 122 pp. M. 2.00

MERLET (P.). See LORMEAU (E.).

METZIG (C.). Auflösungen upd Lösungsanleitungen zu dem Lehrbuche der Arithmetik und Algebra, verbunden mit einer Aufgabensammlung von mehr als 5800 Aufgaben. Berichtigt nach der 3ten Auflage des Lehrbuches. Breslau, Morgenstern, 1909. 8vo. 56 pp. M. 1.60

MINET (A.) et PATIN (L). Cours pratique d'arithmétique, de système métrique, et de géométrie. Cours moyen. 21e édition, revue, corigée et augmentée de problèmes de récapitulation. Paris, Nathan, 1909. 16mo. 314 pp.

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. Ausgabe B: Für reale Anstalten und Reformschulen, unter Mitwirkung von A. Hupe. (H. Müller's mathematisches Unterrichtswerk.) Abteilung II der Oberstufe: Geometrie der Kegelschnitte. Darstellende Geometrie. 3te Auflage. Leipzig, Teubner, 1909. 8vo. 8 + 178 pp. M. 2.40

NABER (H. A.). Das Theorem des Pythagoras, wiederhergestellt in seiner ursprünglichen Form und betrachtet als Grundlage der ganzen Pythagoräischen Philosophie. Haarlem, 1909. 8vo. M. 7.00

PATIN (L.). See MINET (A.).

PINCHERLE (S.). Algebra elementare. 10a edizione riveduta. Milano, Hoepli, 1909. 16mo. 8 + 210 pp.

SALOMON (A.). Nouvelles leçons de géométrie pratique et théorique à l'usage de l'enseignement secondaire des jeunes filles (classes de 3e et 4e années) et de l'enseignement primaire supérieur. 2e édition. Paris, Vuibert, 1909. 16mo. 10 + 302 pp.

SCHUSTER (M.). Geometrische Aufgaben und Lehrbuch der Geometrie: Planimetrie; Stereometrie; ebene und sphärische Trigonometrie. Nach konstruktiv-analytischer Methode bearbeitet. Ausgabe A: Für Vollan-

stalten. 1ter Teil: Planimetrie. 3te Auflage. Leipzig, Teubner, 1908. 8vo. 10 + 158 pp. M. 2 20

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauche an höheren Lehranstalten. Im Sinne der Meraner Lehrpläne bearbeitet. Vol. I: Lehr- und Uebungsbuch für den Unterricht in der Arithmetik und Algebra. 1ter Teil: Für die mittleren Klassen sämtlicher höheren Lehranstalten. 2te, durchgesehene Auflage. Leipzig, Freytag, 1909. 8vo. 203 pp. M. 2.80

SIMON (P.). Traité de géométrie élémentaire à l'usage des élèves de l'enseignement secondaire (premier et second cycles), suivi de compléments à l'usage des candidats aux écoles du gouvernement. 2e édition. Paris, Belin, 1909. 8vo. 451 pp. Fr. 4.00

SOCCI (A.) e TOLOMEI (G.). Aritmetica generale e algebra: libro di testo per i licei. Vol. II, per la 2a classe. Firenze, Le Monnier, 1909. 8vo. 106 pp. L. 1.25

TOLOMEI (G.). See SOCCI (A.).

III. APPLIED MATHEMATICS.

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique élémentaire à l'usage des élèves des classes de mathématiques A et B, conformément aux programmes de 1905. 1re partie: Notions géométriques; Cinématique. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1909. 16mo. 9 + 180 pp. Fr. 2.75

BERNIOLLE (P.). Leçons de géométrie descriptive, conformes aux programmes de 1905. Pour les classes de première C et D. 3e édition. Paris, Paulin, 1909. 18mo. 194 pp. Fr. 2.50

BOCQUET (J. A.). Corso elementare di meccanica applicata, ad uso delle scuole professionali, industriali, tecniche e degli operai. Traduzione di F. Sinigaglia. 2a edizione, riveduta e corretta sulla 5a edizione francese. Napoli, Pellerano, 1909. 8vo. 6 + 359 pp.

BOULANGER (A.). Hydraulique générale. Vol. I: Principes et problèmes fondamentaux. Paris, Doin, 1909. 18mo. 16 + 382 pp.

CAPART (G.). See VIEWEGER (H.).

CHAPPUIS (J.). See APPELL (P.).

EGERER (H.). Neue Methoden der Berechnung ebener und räumlicher Fachwerke. Berlin, Springer, 1909. 8vo. 7 + 96 pp. M. 2.40

GUAZZUGLI (M. G.). Nozioni fondamentali d'idrologia. Roma, Armani, 1909. 8vo. 104 pp.

HAEDER (H.). Konstruieren und Rechnen. Für Studium und Praxis bearbeitet. 4te, erweiterte Auflage. 2 vols. Wiesbaden, Haeder, 1909.
1. Maschinenelemente. 8vo. 16 + 896 pp. Cloth. M. 12.50
2. Beispiele und Tafeln. 11 + 296 pp. Cloth. M. 15.00

JANTZEN (J.). Berechnung und Konstruktion der Einspritz-Kondensatoren und Luftpumpen. Lehrbuch für Studierende des Maschinenbaues und gleichzeitig ein Handbuch für ausübende Techniker und Ingenieure. Hannover, Jänecke, 1909. 8vo. 4 + 113 pp. Cloth. M. 6.00

JOUGUET (E.). Lectures de mécanique. La mécanique enseignée par les auteurs originaux. 2e partie: L'organisation de la mécanique. Paris, Gauthier-Villars, 1909. 8vo. 290 pp. Fr. 10.00

KREUTER (F.). Beitrag zur Berechnung und Ausführung der Staumauern. München, Oldenbourg, 1909. 8vo. 54 pp. Cloth. M. 2.40

KURZWEIL (L.). Die Theorie des Baumeisters; Mathematik und Festigkeitslehre. Für die Baumeisterprüfung und zum Gebrauche für Architekten, Baumeister und Bautechniker in der Praxis. Wien, Graefer, 1909. 8vo. 8 + 172 pp. Cloth. M. 4.80

LANGE (W.). Sammlung von mathematischen Formeln und bautechnischen mathematisch zu lösenden Aufgaben. Ein Hilfsbuch für Bautechniker und Baugewerkschüler zur Wiederholung und zum Selbstunterricht. 3te Auflage. Bremen, Winter, 1909. 8vo. 107 pp. Cloth. M. 3.50

RICHARD (G.). Mécanique, revue. 31e édition. Paris, Dunod, 1909. 16 mo. 5 + 229 + 61 pp. Fr. 2.50

SINIGAGLIA (F.). See BOCQUET (J. A.).

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